Classical mechanics
Summer 2013

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November 21, 2014

Beginning date: May 30, 2012

Abstract

The main objective of this document is to outline my understanding of the reformed classical mechanics in terms of the principle of *least action* and Hamilton and Lagrange's equations. These notes are based on mainly Professor Susskind's lectures from Stanford at iTunes U and a few other resources. I would highly recommend them to anyone looking to understand the concepts of the subject.

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1 Introduction to classical mechanics

Classical mechanics is the basis of all of physics, not because it describes the motion of particles and mechanical system, but because the structure of classical mechanics carries over into all of physics, which include concepts such as the conservation of energy and momentum.

1.1 A-level mechanics: Newtonian mechanics

The physics I was tought at A-level, involved mechanics in the form of Newtons three laws of motion, I am sure anyone reading[1] will remember these laws but I will list them below anyway so that they are easy to refernce if needed in the document[2]:

. *Law I: Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.*

  Or more simply, an object remains in the same motion unless acted upon by an external force. This is law introduces the concept of *inertia*, which can be understood as the tendency of objects to resist changes in motion. Infact Newton was not the first person to postulate this concept, Aristotle has the view that all objects have a natural place in the universe (I shall describe his views later on) and more recently Galileo put forward the concept of inertia and was the first one to correctly formalise it (indeed Newton gave credit to Galileo for this law and it is sometimes called the *Law of inertia*).

. *Law II: The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that impulse is impressed.*

  Or the most familiar law ever[3]:

$$\mathbf{F} = m\mathbf{a} \tag{1.1.1}$$

. *Law III: To every action there is always an equal and opposite reaction: or the forces of two bodies on each other are always equal and are directed in opposite directions.*

We shall see that these laws lead to conservation laws that apply throughout physics. Firstly we set out the logic behind classical physics.

[1] which will be me most of the time!

[2] to make it look more classy then the high school definitions I have just got the literal translations from the principia that are given on wikipedia

[3] other than maybe $E = mc^2$

1.2 Systems, states and laws of motion

Lets begin with the simplest system we can think off. An assumption we make is that time evolves naturally and it can be any real number, and it only occurs in discrete intervals. Now imagine systems that evolve in this time and only have a small set of configurations (states).

Phase space

Sometimes also known as the space of states, is a mathematical space (e.g Hilbert spaces) that contains all possible states of the system. A state can be defined by all the information that one needs about the state to describe how it will change from one instance to the next.

Two-state system

Suppose there is a system which only has two configuration(The phase space will contain only two points as there are only two states), like a dice. So the two possible states are:

$$\text{Heads} = H$$

$$\text{Tails} = T$$

Now we want add a laws of motion (LOM) for this two-state system which describes how it will evolve in time.

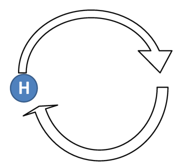
LOM 1

A possible LOM for this system could be that when the system is in a given state, it stays in the same state in the next interval of time:

$$H \rightarrow H$$

$$T \rightarrow T$$

Or equally on a map of states it would look like a loop diagram:

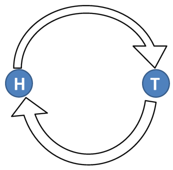


LOM 2

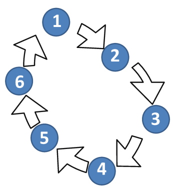
Another LOM for this system could be that the system simply alternates between the two states:

$$H \to T \to H \to T$$

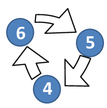
The mapping would look like:

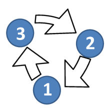


The important point about these laws is that they are deterministic, such that if one knows what happens at a given instance, one can predict what will happen next and infinitly into the future. To generalise this we could consider a system with more than two states. We could consider a die as another example, ofcourse it has 6 sides, therefore 6 states. Now we have a large variety of possible laws for this system such as :



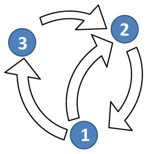
This law cycles through all the states, however we could have laws like:





These mappings show clear difference between this law and the previous one, as this contains *disconnected* cycles. These are equally acceptable laws of physics; as they are completely deterministic.

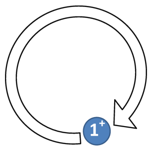
Now consider the LOM:

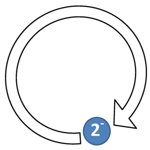


This LOM is not allowed in classical mechanics as it is clearly non-deterministic (at any given state, we do not know *uniquely* where to go next). As a rule of thumb, we can say that for an acceptable LOM there should be only one arrow coming in and one going out of a state, for a given mapping.

1.3 Conservation laws

Conservation laws come about when we have disconnected cycles (as shown above), and they basicly state that some piece of knowledge about a system is kept intact and does not change in time. As an example consider a two state system with conserved quantities at each state, say $^+$ for state one and $^-$ for state two:





In this case these arbitrary variables $^+$, $^-$ are conserved as the LOM shows that once you are in a state with one of these variables you always stay with that variable. So we see that the conservation laws are associated with these closed trajectories in phase space. This can also be called *information conservation*, as we always keep track of where we started and this is the most fundamental concept of classical physics.

Now suppose that for our two state system has a LOM that goes as follows:

$$HH \to H$$

$$HT \to H$$

$$TH \to T$$

$$TT \to T$$

To explain the notation, the fact that therea are two letters means that the previous two states are given, i.e the HH shows that the previous two states were heads and so on. So if the only information that I have is that I am at H then I will not know where to go next, I would need to know what the state was before, (if it was H then I stay at heads if it was T then I go to tails). This is fundamentally different from the other laws, as we need two pieces of information, so far we have only needed one piece of information.

In reality if we want to predict the motion of a particle, ifinitely into the future we need to know the exact location of the particle and its velocity, so it is very much like the law which we made up above. Of course we know that everything cannot be predicted infinitely into the future (just look at the british whether!) [4], the reason for this finite range of predictability is that there is always uncertainty in any measurment one makes. We might say we know the position and velocity of a particle but the precision of our knowledge always catches up with us. To be able to predict infinitely into the future we would need to know this information *exactly.*

This means that the phase space will have two axis (as supposed to one as we had in the systems described in the previoue section), the mathematical origin of the two axis actually arrises from the LOM being Newton's equations and they happen to be *second order differential equations*:

$$\mathbf{F}(\mathbf{x}) = m\mathbf{a} = m\frac{d^2\mathbf{x}}{dt^2} = m\ddot{\mathbf{x}} \tag{1.3.1}$$

So if we know the position we can get the acceleration (as force depends on position[5]), but there is no way to obtain the velocity so it has to be added in as an extra piece of information.

1.4 Aristotle vs Newton

Before Netwon, greek philosophor Aristotle postulated this law:

$$\mathbf{F}(\mathbf{x}) = m\mathbf{v} = m\dot{\mathbf{x}} \tag{1.4.1}$$

[4]these leads to a very interesting philosophical arguement that I might include as a side as it has puzzled me for years now

[5]as an analogy we can think of Coloumb force or gravitation force

This shows that without any force we cannot have velocity, or in other words, objects with no force acting on them will remain *stationary* (notice the subtelty from Newton's second law, where an object with no force, will not necessarily remain stationary, but if it is already travelling at a velocity it will continue to do so). And once again there is the usual assumption that we know what the force is when we know the position. This law would mean that the only piece of information we need would be the position of the particle. As knowing the poistion would give the force and then we can just read of the velocity. We can also get the acceleration from the equation by simply differentiating the velocity w.r.t time:

$$\frac{d\mathbf{F}(\mathbf{x})}{dt} = m\frac{d^2\mathbf{x}}{dt^2} = m\mathbf{a} \tag{1.4.2}$$

Now we may ask why would the force on the particle change with time and the reason is that the particle will move under the influence of the force and therefore the position will change and therefore the force will also change. Using the chain rule we can write:

$$\frac{d\mathbf{F}}{dt} = \frac{d\mathbf{F}}{d\mathbf{x}}\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{F}}{d\mathbf{x}}\mathbf{v} \tag{1.4.3}$$

Now we substitute this into 1.4.2:

$$m\mathbf{a} = \frac{d\mathbf{F}}{d\mathbf{x}}\mathbf{v} \tag{1.4.4}$$

So without any further information we can deduce the acceleration aswell by differentiating the force w.r.t position (which I assuming that we know the force as a function of position). We could differentiate this equation w.r.t time again and obtain the derivative of the acceleration (third moment of position), sometimes known as *Jerk*:

$$m\dot{\mathbf{a}} = \frac{d\mathbf{F}}{d\mathbf{x}}\mathbf{a} + \frac{d^2\mathbf{F}}{d\mathbf{x}^2}\mathbf{v}^2 \tag{1.4.5}$$

Here I have just used the product rule, and we see that we can also calulate the jerk and we can carry on for as many moments as we like. Therefore this law is completely deterministic and only requires one piece of information. But this is not how nature works, we know the reality is a little bit different.

<u>Newton's law</u>

Not all the different from Aristotle's law, in appearance, is Newton's second law:

$$\mathbf{F}(\mathbf{x}) = m\mathbf{a} = m\ddot{x} \tag{1.4.6}$$

In this case if one knows the position, the force is known as it depends on position and the acceleration can the be read off. Now if we want to calculate the velocity we would need to integrate the acceleration, which would mean integrating the force w.r.t position. Ofcourse there is no problem in doing this, but integrating without limits will give atleast one unkown constant term. To get rid of it we would need to know the limits which would mean another position or atleast another piece of information (we generally want the initial position and velocity. Of course knowing the velocity is just the same as saying that we know what the posistion is now and at a previous time).

Basically there is no way of calculating the velocity without any extra information. Hence we have to add the velocity as an extra piece of information to the initial condidtions. Once this is known the law is completely deterministic and we can predict the configuration of the system at any given time in the future.

1.5 Energy conservation

The concept of energy conservation has been viewed experimentally in every physical experiment and here I shall show how it comes about from Newton's laws (I must stress that it is infact a much deeper concept and holds in all parts of physics).

As I say this, firstly I will give an example of a hypothetical system in which energy is not conserved. Suppose there is a force law that always pushes a particle around in a cricle, so the particle will increase in its velocity. So at the same position in the circle the potential energy will be the same, as it depends only on the distance from the particle (and the radius is always the same!), but notice that its kinetic energy has increase. Even though this LOM is completely deterministic it is never observed as it violates the conservation of energy.

In nature, forces are always conservative and of the form:

$$\mathbf{F} = -\nabla U(\mathbf{x}, \mathbf{y}) \tag{1.5.1}$$

This basicly states that the force in a given direction is proportional to the change in potential energy in that direction. The total energy is the sum of the potential energy,U, and the kintec energy,T:

$$E = U + T = U + \frac{1}{2}m\mathbf{v}^2 \tag{1.5.2}$$

Now we want to prove that energy is conserved as a consequence of Newton's equations.

The time derivative of kinetic energy:

$$\begin{aligned} \frac{dT}{dt} &= \frac{1}{2}m\mathbf{v}\frac{d\mathbf{v}}{dt}2 \\ &= m\mathbf{v}\frac{d\mathbf{v}}{dt} \\ &= m\mathbf{v}\mathbf{a} \end{aligned} \tag{1.5.3}$$

The time derivative of the potential energy is:

$$\begin{aligned} \frac{U(\mathbf{x})}{dt} &= \frac{\partial U(\mathbf{x})}{\partial x}\frac{\partial \mathbf{x}}{\partial t} \\ &= \frac{\partial U}{\partial \mathbf{x}}\mathbf{v} \end{aligned} \tag{1.5.4}$$

Now putting these together we can compute the time derivative of the total energy:

$$\begin{aligned} \frac{\partial E}{\partial t} &= \frac{\partial T}{\partial t} + \frac{\partial U}{\partial t} \\ &= \mathbf{v}m\mathbf{a} + \frac{\partial U}{\partial \mathbf{x}}\mathbf{v} \end{aligned} \tag{1.5.5}$$

But according to Newton's laws:

$$\frac{dU}{dx} = -F = -m\mathbf{a} \tag{1.5.6}$$

Substituting this into the previous equation of the derivative of the total energy we get:

$$\frac{dE}{dt} = 0 \tag{1.5.7}$$

So the total energy is conserved.

1.6 Momentum conservation

We are still talking about a single particle. Lets rewrite Newton's equation as:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} \tag{1.6.1}$$

We are assuming that mass does not change with time, we can bring it inside the derivative:

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt} \tag{1.6.2}$$

So for the momentum to be conserved the force has to be zero and this is the form of Newton's second law that we shall mostly use from now on.

Newton said that the force on every object is the sum of forces due to all the other objects and forces between the particles are equal and opposite. We want to show that the time derivative of the total momentum is zero. Suppose we have three particles in our system; $1, 2, 3$. The equations we get for the force are as follows:

$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{1,2} + \mathbf{F}_{1,3} \tag{1.6.3}$$

To clarify the notation, this equation is for particle 1 with momentum p_1.
$F_{1,2}$ represents the force on 1 due to 2
$F_{1,3}$ represents the force on 1 due to 3
We have also assumed that the particle exerts no force on itself! (this is infact a statement that the particle is not extended, it is point-like).

Similarly we have equations for the other particles:

$$\frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{2,1} + \mathbf{F}_{2,3} \tag{1.6.4}$$

$$\frac{d\mathbf{p}_3}{dt} = \mathbf{F}_{3,2} + \mathbf{F}_{3,1} \tag{1.6.5}$$

The total force is:

$$\frac{d\mathbf{p}_T}{dt} = \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} + \frac{d\mathbf{p}_3}{dt} \tag{1.6.6}$$

Now we can substitute in the forces to get:

$$\frac{d\mathbf{p}_T}{dt} = \mathbf{F}_{1,2} + \mathbf{F}_{1,3} + \mathbf{F}_{2,1} + \mathbf{F}_{2,3} + \mathbf{F}_{3,1} + \mathbf{F}_{3,2} \tag{1.6.7}$$

But from Newton's 3^{rd} law we know that:

$$\begin{aligned} \mathbf{F}_{1,2} &= -\mathbf{F}_{2,1} \\ \mathbf{F}_{1,3} &= -\mathbf{F}_{1,3} \\ \mathbf{F}_{2,3} &= -\mathbf{F}_{2,3} \end{aligned} \tag{1.6.8}$$

and so on... Therefore we are simply left with:

$$\frac{d\mathbf{P}_T}{dt} = 0 \tag{1.6.9}$$

Showing the fact that momentum is conserved.

2 Lagrangian formulation and the principle of least action

All of the things discussed were part of Newtonian physics that we are (or atleast I was), taught at high school/ASfirst year level. However these laws were written in a more general form that reveal the deeper concept of action in physics. However before we get into the physics, I will briefly go through some mathematical concepts that we will need.

2.1 Review: Integration by parts

Suppose we have:

$$\int_{t_1}^{t_2} dt \frac{dF}{dt} \tag{2.1.1}$$

If this was an indefinite integral it would have just returned the function F itself, but because its between two limits, it just returns the value of the function at two different times:

$$\int_{t_1}^{t_2} dt \frac{dF}{dt} = F(t_2) - F(t_1) \tag{2.1.2}$$

We can think of it by examining dF carefully. It basicly means infinitesimal changes in the function:

$$dF = F(2) - F(1) \tag{2.1.3}$$

where $F(2)$ represents the function at point 2 and $F(1)$ represents the function at point 1 anf the points themselves are infinitesimally close; therefore in general:

$$dF = \{F(2) - F(1)\} + \{F(3) - F(2)\} + \{F(4) - F(3)\}...\{F(n) - F(n-1)\}$$

The $dt's$ are similar time intervals and since they are in the numerator and denominator they just cancel. So we get:

$$dF = F(n) - F(1) \tag{2.1.4}$$

As the rest of the terms just cancel.
Now suppose F is a function of two functions:

$$F = f(t)g(t)$$

And in the integral we have $\frac{dF}{dt}$, so now we have to use the chain rule:

$$\frac{dF}{dt} = \frac{df(t)}{dt}g(t) + \frac{dg(t)}{dt}d(t) = \dot{f}g + \dot{g}f \tag{2.1.5}$$

therefore the integral 2.1.2 becomes:

$$\int_{t_1}^{t_2} \left(\dot{f}g + f\dot{g}\right) dt \tag{2.1.6}$$

We know that the integral has a solution of:

$$F(n) - F(1) = F(t)|_{t_1}^{t_2} \tag{2.1.7}$$

Now lets consider the special case in which the r.h.s of the equation above is zero:

$$fg|_{t_1}^{t_2} = 0$$

Then we can write:

$$\int_{t_1}^{t_2} dt\dot{f}g = -\int_{t_1}^{t_2} dt\dot{g}f \tag{2.1.8}$$

So we see that one can readily change which function is differentiable by changing the sign as long as the boundary term is zero.

2.2 Minimising functions

Suppose we have a function $f(y)$, that looks like

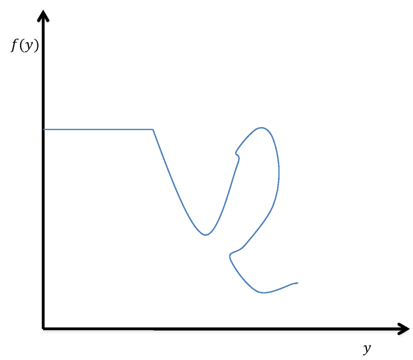


Figure 1: The function $f(y)$

At a minimum of the function we have the condition:

$$\frac{df(y)}{dy} = 0 \tag{2.2.1}$$

A *local* minimum is a minimum in a confined region.
A *global* minimum is the lowest minimum in the entire function.

The basic problem of mechanics is to determine the trajectory of a system from its initial conditions. Particles have coordinates which describe their positions, we can label them as $q_1, q_2, q_3, (x, y, z)$, in general we could have n number of coordinates. But we have already seen that the $q's$ are not enough to predict the future, we also need the velocities, $\dot{q}'s$. To generalise we need $6n$ pieces of information to predict the motion of n particles.

By predicting the motion, I mean that we want to find out $q_1(t), q_2(t), q_3(t)$ and so on. These variables as a function of time can be called trajectories (so a set of $q's$ at every value of time)

2.3 Calculus of variations

The principle of least action reformulates the equations of motion given by Netwon. The two pieces of information in this case are the beginning and end points of the trajectory. This information is sufficient to tell us which trajectory the system will take in phase space and the trajectory is *uniquely* determined by the principle of least action. This tells us that there is some quantity that is associated with this whole trajectory which is minimised for the unique trajectory taken by the system.

Consider two points in phase space, connected by a random trajectory. Now suppose we want the curve to take up the smallest possible length between the two points. The answer is obviously a straight line between the points, but this can be tough to show mathematically (as we would have to consider every possible trajectory!). An easier way to see this is to zoom in on one part of the curve

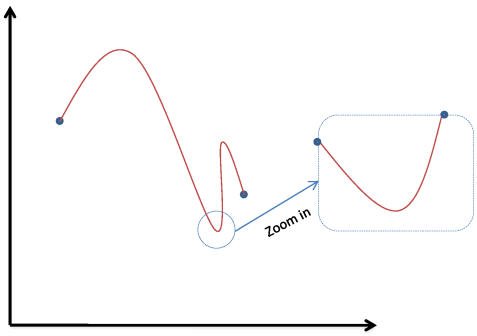


Figure 2: A random trajectory connecting two points in phase space

The curve can be thought of as being made up of many of these curves. Lets ask what would be the minimised trajectory on these smaller curves. The answer does not depend on the beginning and final conditions of the system, it instead depedns on the beginning and final conditions of the part we have zoomed in on. The answer is obvious, that we should have the points connected by a straight line. Now this is easy to generalise and say, just go ahead at every point in the trajectory and we get a straight line.
To formulate this problem mathematically, consider again an infinitesimal section of a curve in the $x-y$ plane

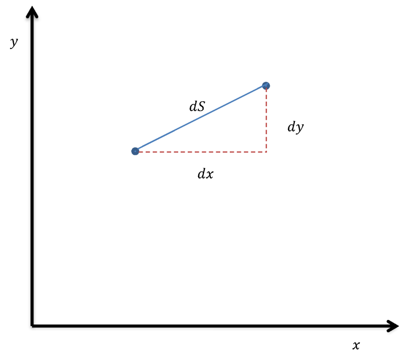


Figure 3: Infinitesimal part of a curve

In this case y is the function and x is the independent variable. The curve goes through points x_1 and x_2, these are the starting and ending points and we want to find the least distance between the two points. dS is just the small length along the curve and is given by:

$$dS = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{2.3.1}$$

Therefore the overall length of the curve is given by the integral of dS:

$$S_{1,2} = \int_{x_1}^{x_2} dS = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{2.3.2}$$

Now the problem we face is making $\left(\frac{dy}{dx}\right)$ as small as possible to minimise the integral $S_{1,2}$. So one might naturally say that we should make $\left(\frac{dy}{dx}\right) = 0$, but we cannot do that as that mean the gradient of the curve is zero, which is obviously not the case in this case. So here we have defined mathematically what the problem is; we have to find the function $y(x)$, that minimises the expression.

The S here, depends on $y(x)$, we can think of it as a function of a function and such a quantity is usually called a *functional*. The result is still a number, but the input is a function. Examples of such problems include *Fermats principle of least time* in optics, that states that light rays travel in paths that minimise the time taken to travel to a given point (as supposed to taking the path that is the smallest). Mathematically these problems are called *calculus of variations* (and the principle of least action comes from this).

2.4 Principle of least action

The *action*, A, is defined as:

$$A = \int (T - U)dt \tag{2.4.1}$$

Where T generally stands for kinetic enegry and U is the potential enegry and they take the general form of:

$$T = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}m\left(\frac{d\mathbf{x}}{dt}\right)^2$$

$$U = U(\mathbf{x}) \quad \text{So its a function of position}$$

Therefore the action can be written as:

$$A = \int \left(\frac{1}{2}m\left(\frac{d\mathbf{x}}{dt}\right)^2 - U(\mathbf{x})\right)\, dt \tag{2.4.2}$$

This may seem a strange defination, but the only way to test weather it is true or not, is to test it for all classical system and see that it reproduces Newton's equations.

Lets compare it to the principle of least time in optics and the least distance that I described in the graphs previously. In the previous case we had x as the independent variable now we are using time as the independent variable. The thing inside the integral is called the *Lagrangian* and in general it is a function of the coordinates $q's$ and the derivative of the coordinates $\dot{q}'s$:

$$L = L(q_i, \dot{q}_i) \tag{2.4.3}$$

Hence the action, in its most general form, is:

$$A = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)dt \tag{2.4.4}$$

All of the laws of classical physics come from this principle of least action. A counter example one could think of are the laws of thermodynamics as they come from statistical principles, however the statistics itself describe a large number of degrees of freedom (d.o.f) which in itself follow the principle of least action.

In physics, systems can be described by *local* or *global* LOM. A local LOM is one that describes how to move in the next instance at a given point in the trajectory (such as Newton's second law)

A global LOM is one that depends upon the entire trajectory of the system (not on any given point in the trajectory) and principle of least action (PLA) is an example of this. It states that a trajectory in phase space has a quantity associated with it known as the action that is minimised. To do this calculation we need to know the $q's$ and $\dot{q}'s$ at the beginning and end of the trajectory.

Suppose we have a trajectory that is described by the dynamical variables:

$$\hat{q}_i(t) = \text{Trajectory 1} \tag{2.4.5}$$

The hat is there to show that this is the true trajectory of the system. Now if we change the trajectory:

$$\hat{q}_i(t) + \alpha f_i(t) = \text{Trajectory 2} \tag{2.4.6}$$

where $f_i(t)$ is a subset of functions and α is just a number. The important point is that we force the trajectory to pass through the initial and final points that correspond to $\hat{q}_i(t)$. Lets calulate the action for the trajectories. For trajectory 1 A is a function of the $\hat{q}_i{}'s$. For trajectory two it is a function of α if we are given q_i and α.
For the true trajectory that corresponds to the PLA is the trajectory which has $\alpha = 0$. Therefore we can also say:

$$\frac{\partial A(\alpha)}{\partial \alpha} = 0 \quad \text{when } \ \alpha = 0 \tag{2.4.7}$$

Therefore the function $A(\alpha)$ has a stationary point at $\alpha = 0$. The action was defined as:

$$A = \int_{t_1}^{t_2} dt L(q(t)\dot{q}(t))$$

Lets also define the derivatives:

$$\frac{dq_i}{d\alpha} = f_i(t)$$

$$\frac{d\dot{q}_i}{d\alpha} = \dot{f}_i(t)$$

Now lets see how much the action changes:

$$\frac{dA}{d\alpha} = \int_{t_1}^{t_2} dt \frac{\partial L}{\partial q_i}\frac{dq_i}{d\alpha} = \int_{t_1}^{t_2} dt \frac{\partial L}{\partial q_i} f_i(t) \tag{2.4.8}$$

The Lagrangian changes a small amount as the $q's$ and the $\dot{q}'s$ will change a little as α changes. But there is also another term that we have to add as there is a change in the Lagrangian as $\dot{q}$ changes:

$$\frac{\partial A}{\partial \alpha} = \int_{t_1}^{t_2} dt \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}_i}{d\alpha} = \int_{t_1}^{t_2} dt \frac{\partial L}{\partial \dot{q}_i} \dot{f_i(t)} \tag{2.4.9}$$

So the overall derivative of the action w.r.t α is just the sum of the two terms:

$$\frac{\partial A}{\partial \alpha} = \int_{t_1}^{t_2} dt \sum_i \left(\frac{\partial L}{\partial q_i} f_i(t) + \frac{\partial L}{\partial \dot{q}_i} \dot{f}_i(t) \right) \tag{2.4.10}$$

Remember that $f_i(t)$ can be any function as long as it goes to zero at the boundaries. But we have a $\dot{f}_i(t)$ in the previous equation as well, which we would prefer to be just $f_i(t)$ and we can obtain this by using integration by parts:

$$\frac{\partial A}{\partial \alpha} = \int_{t_1}^{t_2} dt \sum_i \left(\frac{\partial L}{\partial q_i} f_i - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) f_i \right) \tag{2.4.11}$$

Note that the boundary term is zero, as the functions were stipulated to go through the boundary points and to be zero at those positions. So now we set $\frac{\partial A}{\partial \alpha}$ as is the requirement for minimising action:

$$\int_{t_1}^{t_2} dt \sum_i \left(\frac{\partial L}{\partial q_i} f_i - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) f_i \right) = 0 \tag{2.4.12}$$

The f_i can now be taken outside the brackets:

$$\int_{t_1}^{t_2} dt \sum_i \left(\frac{\partial L}{\partial q_i} - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \right) f_i = 0 \tag{2.4.13}$$

But we know that this has to work for any function f_i (as long as it is zero at the boundary), therefore the term inside the bracket must be zero:

$$\frac{\partial L}{\partial q_i} - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{2.4.14}$$

This has the form of a local equation as it states that at each point in the trajectory the action has to be minimised. This is called the *Euler-Lagrange*(E-L) equation. It is the most fundamental equation in all of physics.

Now we introduce some new notation:

$$\pi_i = \frac{\partial L}{\partial \dot{q}_i} = \text{Conjugate momentum} \tag{2.4.15}$$

This is a momentum corresponding to a posistion component q_i (also known as the canonical momentum conjugate).

$$\frac{\partial L}{\partial q_i} = \text{Generalised component of force} \tag{2.4.16}$$

The principle of least action is not literally correct as we only require that $dS = 0$. This means that S could be minimised, maximised or be a saddle point. Since $L = T - U$, we can always increase S by taking having a high kinetic energy and so the true path is never a maximum. However, it may be either a minimum or a saddle point, this fact is sometimes known as "'Hamilton's principle"'.

All the fundamental laws of physics can be written in terms of the principle of least action[6].

[6]Infact (almost) everything we know is described by the Lagrangian, $L = \sqrt{g}(R - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\not{D}\psi)$

2.5 Examples

To cement the understanding of these new principles, we shall now do a few examples that should be familiar.

Example 1: Single particle in 1D

The kinetic energy is:

$$T = \frac{1}{2}m\dot{x}^2 \tag{2.5.1}$$

Potential energy:

$$U = U(x) \tag{2.5.2}$$

Therefore the Lagrangian:

$$L = \frac{1}{2}m\dot{x}^2 - U(x) \tag{2.5.3}$$

In this case the dynamical variables that were used to generalise the framework, $q's$, are replaced by x. Now we simply substitute this Lagrangian into the E-L equation:

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x} = p \tag{2.5.4}$$

Here p has taken the place of π to show that this is simply the momentum we are all familiar with since kinder garden. The other part of the E-L equation is simply the derivative of the potential w.r.t to posistion as the kinetic enegry term has no posistion dependance, so we arrive at the final result:

$$\frac{dp}{dt} = -\frac{dU(x)}{dx} \tag{2.5.5}$$

Which is simply Netwon's second law!

Notice that this is why we use the Lagrangian as $T - U$ as if it were $T + U$ the sign in the equation above would have been different to what was expected from Newton's second law. The Lagrangian is not, in general, a conserved quantity. If we have a large system with many particles in 3D the kinetic energy is simply the sum of all the particles in the three dimensions.

Example 2: 2 particles in 1D with potential $U(x)$

$$\begin{aligned} \text{Particle 1 : position} &= x_1 \\ \text{mass} &= m_1 \\ \text{Particle 2 : posistion} &= x_2 \\ \text{mass} &= m_2 \end{aligned} \tag{2.5.6}$$

For this system we assume *translational* invariance. That means the potential does not depend on where the particles are, it just depends on the distance between them (also known as translation symmetry). In other words; the system does not change if all the components in the field are shifted by the same amount.

The Lagrangian is:

$$L = \frac{1}{2}m_1\mathbf{x}_1^2 + \frac{1}{2}m_2\mathbf{x}_2^2 - U(\mathbf{x}_1 - \mathbf{x}_2) \tag{2.5.7}$$

Now we compute the E-L equation:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\mathbf{x}}_1} = -\frac{\partial L}{\partial \mathbf{x}_1} \tag{2.5.8}$$

But:

$$\frac{\partial L}{\partial \dot{\mathbf{x}}_1} = \mathbf{p}_1 \tag{2.5.9}$$

$$\frac{\partial \mathbf{p}_1}{\partial t} = -\frac{\partial U(\mathbf{x}_1 - \mathbf{x}_2)}{\partial x_1} \tag{2.5.10}$$

And similarly:

$$\frac{\partial \mathbf{p}_2}{\partial t} = -\frac{\partial U(\mathbf{x}_1 - \mathbf{x}_2)}{\partial x_2} \tag{2.5.11}$$

Suppose we relabeled $(\mathbf{x}_1 - \mathbf{x}_2)$ with D:

$$U(\mathbf{x}_1 - \mathbf{x}_2) = U(D)$$

Now we can cleverly manipulate this:

$$\frac{\partial U}{\partial x_1} = \frac{dU}{dD}\frac{dD}{dx_1}$$

$$\frac{dD}{dx_1} = 1$$

Similarly:

$$\frac{\partial U}{\partial x_2} = \frac{dU}{dD}\frac{dD}{dx_2}$$

$$\frac{dD}{dx_2} = -1$$

So we finally get:

$$\frac{\partial U}{\partial x_1} = -\frac{\partial U}{\partial x_2} \tag{2.5.12}$$

Which is equivalent to:

$$\frac{dp_1}{dt} = -\frac{dp_2}{dt} \tag{2.5.13}$$

Which is just a statement of Newton's third law. But if we rewrite this as:

$$\frac{dp_1}{dt} + \frac{dp_2}{dt} = \frac{dp_T}{dt} = 0 \tag{2.5.14}$$

were p_T is the total momentum. Now this is statement of the conservation of momentum. So we see that the conservation of momentum comes about due to translation symmetries (the fact that the potential only depends on the distance between two points and not there position in space. This is an example of a much deeper relation between symmetries and conservation laws as we shall see later.

Example 3: Particle in 2D near the ground

So now there are 2 $q's$ corresponding to the two coordinates (vertical,y, and horizontal,x). The energies are:

$$T = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{1}{2}m\dot{\mathbf{y}}^2 \tag{2.5.15}$$

$$U = \mathbf{y}mg \tag{2.5.16}$$

So the forces are:

$$\frac{\partial U}{\partial x} = 0 \tag{2.5.17}$$

$$\frac{\partial U}{\partial y} = mg \tag{2.5.18}$$

Hence $F = -mg$ represents an attractive force (as expexted!).
The Lagrangian for the system is:

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{1}{2}m\dot{\mathbf{y}}^2 - mg\mathbf{y} \tag{2.5.19}$$

The problem has a translation symmetry in the x direction but not the y direction as a change in y will change the Lagrangian. Note that we can always add a constant into the potential energy and that doesnt change the motion as the force is given by the derivative of the potential.

So we expecet to find one conservation law. Once again we compute the E-L equation:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial t}m\dot{\mathbf{x}} \tag{2.5.20}$$

$$\frac{\partial L}{\partial x} = \frac{\partial(mg\mathbf{y})}{\partial x} = 0 \tag{2.5.21}$$

Therefore:

$$\frac{d}{dt}m\dot{\mathbf{x}} = 0 = \frac{d\dot{p}_x}{dt} \tag{2.5.22}$$

Therefore the momentum in the x direction is conserved. Now doing the same for the y component:

$$\frac{d}{dt}p_y = -\frac{d(mg\mathbf{y})}{dy} = -mg \tag{2.5.23}$$

This just tells us that the vertical acceleration is proportional to the force of gravity, but the y component of the momentum is not conserved as was expected due to the fact that there wass no symmetry associated with it.

<u>Example 4: Particle moving in a plane (circular coordinates)</u>

Instead of using x and y we shall r (radius) and θ (angle from x-axis) to describe the position. The velocity of the particle is given by:

$$v_r = \dot{r} = \text{Radial/tangential velocity} \tag{2.5.24}$$

$$v_\theta = \dot{\theta}r = \text{Angular velocity} \tag{2.5.25}$$

So the kinetric energy is:

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \tag{2.5.26}$$

Suppose the potential energy of the system is coming from a central force. A central force means the potential energy only depends on distance, so in this case it would only depend on r. This means the system has rotational symmetry. So the Lagrangian is:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(\mathbf{r}) \tag{2.5.27}$$

Now we shall compute the components of the E-L equation:

$$\pi_i = \frac{\partial \pi_i}{\partial t} = \frac{\partial L}{\partial \dot{q}_i} = m\dot{r} \tag{2.5.28}$$

where $\dot{q}_i = \dot{r}$. The E-L equation is then:

$$\frac{\partial \pi_i}{\partial t} = \frac{\partial L}{\partial q_i} \tag{2.5.29}$$

where $q_i = r$, so we get:

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial U}{\partial r} \tag{2.5.30}$$

Look at this closely, it is very similar to Newtons second law except for it has an extra term $mr\dot{\theta}^2$. First thing to notice about this term is that it is positive. We can think of it as an extra force that is acting radially outward. This is called (as im sure everyone knows) the centrifugal force, which has the apparent effect of creating repulsion away from the center of rotation.

Now lets do the same for the other dynamical variable, θ:

$$\pi_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{2.5.31}$$

The E-L equation is:

$$\frac{d}{dt}mr^2\dot{\theta} = \frac{\partial L}{\partial \theta} \tag{2.5.32}$$

But

$$\frac{\partial L}{\partial \theta} = 0$$

So we get:

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{2.5.33}$$

We know $mr^2\dot{\theta}$ is called the angular momentum $\mathbf{L}$. So the equation above is a statement that $\mathbf{L}$ does not change with time and is therefore a conserved quantity. The angular momentum is given by the initial conditions and remains constant there-after in this system, so we get:

$$\dot{\theta} = \frac{\mathbf{L}}{mr^2} \tag{2.5.34}$$

Now we can substitute $\dot{\theta}$ into the $E - L$ equation we got for r:

$$m\ddot{r} = -\frac{\partial U}{\partial r} + \frac{\mathbf{L}^2}{mr^3} \tag{2.5.35}$$

The main reason for me to rewrite the radial E-L equation like this is so that we can see that the centrifugal force component goes as $\frac{1}{r^3}$ and the rest of the forces are usually quadratic (like gravity and E-M) which means that the centrifugal forces are much more significant at smaller distance (less than 1) then the other forces.

2.6 New notation

Suppose we have a function:

$$F(\alpha_i)$$

So it depends on a bunch of variables α_i. For the function to be minimised the differentiated function w.r.t all the variables must be zero:

$$\frac{F(\alpha_i)}{\partial \alpha_1} = \frac{F(\alpha_i)}{\partial \alpha_2} = \frac{F(\alpha_i)}{\partial \alpha_3}.... = 0 \tag{2.6.1}$$

This is usually denoted in shorthand by:

$$\delta F = 0 \tag{2.6.2}$$

And it just means that if you are at the minimum of a the function F then a small change in all of the $\alpha' s$ does not change the function to *first order*.

3 Symmetries and conservation laws

The idea of a symmetry is a change that you can make, that does not affect the action. For example, if we have a system of particles moving around and the particles are interacting with each other but not an external object, then if we take the whole system, including its entire motion and shift it by a specific amount, the action (or equivalently the Lagrangian) does not change. This defines a symmetry under *translation*.

Another example would be taking the system and rotating everything by the same angles and the action not changing. Then the system is said to be invariant under rotational transformations. So in general, a symmetry is defined by the change that one can make in the coordinates that doesn't affect the action. We are particularly interested in infinitesimal symmetries (small changes in the system).

But one can build any change out of many small changes. For example if I want to rotate a system by 180° I can simply rotate the system 180 times by a rotation of 1°. In other words, working to first order. Lets define a transformation of coordinates:

$$q_i = q_i + \epsilon f_i(q) \tag{3.0.3}$$

Where ϵ is a small number and f_i can be any function of the coordinates. The statement of symmetry in this case is that after a transformation of the type described above, the action of a trajectory does not change.

As an example consider the rotation under Cartesian coordinates:

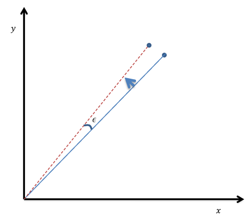


Figure 4: Rotation transformation

Lets define δx as the change in the x coordinate. The ϵ in the diagram represents a small change in the angle:

$$\delta x = -\epsilon y \quad \text{as} \quad sin\epsilon \approx cos\epsilon \approx \epsilon \quad \text{for small angles} \tag{3.0.4}$$

Similarly:

$$\delta y = \epsilon x \tag{3.0.5}$$

The sign simply depends on the direction of rotation. A symmetry would mean that the action does not change after this rotation.

3.1 Noether's Theorem

Consider an axis consisting of two points that are connected by a trajectory, like Fig:2. Lets suppose that this trajectory is a solution to the equations of motion. That means $\delta A = 0$ for all changes of the trajectory as long as the end points are the same. Now lets make a change in the trajectory which *does* change the end points by a symmetry operation (such as moving both points equally in space), but still $\delta A = 0$ since the transformation occurs in a symmetry (and a symmetry is defined as a transformation that does not change the action).
Now lets compute the action after this transformation and see what this leads to. As a reminder, the action is:

$$A = \int L dt \tag{3.1.1}$$

and

$$\delta A = \int \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \tag{3.1.2}$$

Using integration by parts (as shown before), we can rewrite this as:

$$\delta A = \int \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \tag{3.1.3}$$

But now δq_i is not zero at the end points (since the end points have also been shifted), so the boundary term from integration by parts is still there, so we have to add that term:

$$\delta A = \int \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial L}{\partial \dot{q}_i}\delta q_i|_{t_1}^{t_2} \tag{3.1.4}$$

The important thing to realise is that the initial trajectory was a solution to the E-L equation, which means it satisfies:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{3.1.5}$$

And the only thing that is different in this trajectory is the boundary term. So we can infact ignore the first term inside the integral and write:

$$\delta A = \int \frac{\partial L}{\partial \dot{q}}\delta q_i|_{t_1}^{t_2} \tag{3.1.6}$$

But the fact there is a symmetry tells us that $\delta A = 0$, so:

$$\frac{\partial L}{\partial \dot{q}}\delta q_i|_{t_1}^{t_2} = 0 \tag{3.1.7}$$

But the boundary term is the difference between a quantity at two different times. So the fact that the quantity is the same is the same as saying that the quantity is conserved. So we immediately see how a conservation law comes from the symmetry (Note that we have implicitly assumed that we are summing over all coordinates). Therefore the conserved quantity is:

$$\frac{\partial L}{\partial \dot{q}_i} \delta q_i$$

But remember that δq_i can be written as:

$$\delta q_i = \epsilon f_i(q_i) \tag{3.1.8}$$

So the conserved quantity can be written as:

$$\frac{\partial L}{\partial \dot{q}_i} \epsilon f_i(q) \tag{3.1.9}$$

But the fact that this quantity does not change with time means that any constants are irrelevant and we can ignore ϵ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} f_i(q)\right) = 0 \tag{3.1.10}$$

Another way to say it is that $\pi_i(q_i)$ is a conserved quantity and this is called the *Noether* charge.

Example 1: Translation

Set of particles that are invariant under translation. Lets move along the x-axis, δx_i, now is representing particle particle label. It does not label directions of space, so δx_i represents the change in the x coordinate of the ith particle, which is just ϵ:

$$\delta x_i = \epsilon \tag{3.1.11}$$

This just says that you translate by this small amount ϵ wherever you are. Lets keep the symmetry just for the x-axis (of course there could also be a different symmetry along y) and take:

$$\delta z_i = \delta y_i = 0 \tag{3.1.12}$$

In other words all of the $f_i(q_i)$ function that we defined before, are equal to one for the x direction. For the y and z directions they are equal to zero. Now lets calculate the Noether charge, so we sum over all particles, the canonical momentum conjugate to the x coordinate times f:

$$\sum_i \pi_x = \sum_i m_i \dot{x}_i \tag{3.1.13}$$

So we see that is the familiar momentum that is the conserved Noether charge in this case. Notice that we did not need to know anything about the forces, except for the fact that they do change when we move the *whole* system.

Example 2: Rotation

Once again refer to the Fig . One particle moving in a plane, being rotated. Once again we want to calculate the related Noether charge, first remember:

$$\delta x = -\epsilon y = \epsilon f_x$$

$$\delta y = -\epsilon x = \epsilon f_y$$

If ϵ is positive, then the rotation is anti-clockwise. If it is negative it corresponds to a rotation in the clockwise direction:

$$f_x = -y \quad \text{and} \quad f_y = x \tag{3.1.14}$$

Now we can work out the Noether charge:

$$-p_x y + p_y x = (x p_y - y p_x) = \mathbf{L} \tag{3.1.15}$$

Which is simply the z component of the angular momentum. And this holds for all the particles in the system.

Time dependence

A closed system usually has a symmetry under transformations in time, by which I mean the system should be able to go backwards in the same way that it has come forwards in time. For example a planet and a star with no external effects should rotate around their mutual centre of mass no matter at what time we start looking at them.

Suppose again we have a trajectory that is a function of the dynamical variables and that follows the equations of motion. But now the q's depend on time, $q(t)$. Lets move the trajectory forward in time by an amount ϵ (and I mean each point of the trajectory), so that $t \to t + \epsilon$:

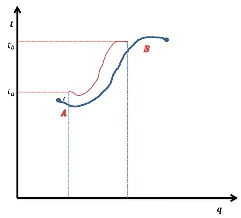


Figure 5: Time transformation

But we know that q now depends on time, so if you like we can also think of the same translation by shifting the q's in a particular way (in this case it would be to the left in phase space). So lets try to formulate what happens to q at each instance of time:

$$\delta q(t) = -\frac{\partial q}{\partial t}\epsilon \tag{3.1.16}$$

The negative sign is there to show that a positive $\frac{\partial q}{\partial t}$ will shift the trajectory backward in phase space. If there is a symmetry, we know that the action of the new trajectory must remain unchanged. Towards the end of the trajectories there are two extra pieces that we have to treat differently (look at the diagram).

Now lets write down the total change in the action between the two trajectories. Firstly lets consider the action between t_a and t_b:

$$\delta A = \int dq \frac{\partial L}{\partial q}(-\dot{q}\epsilon) + \frac{\partial L}{\partial q}\delta\dot{q} \tag{3.1.17}$$

Now we have two extra pieces at the ends which I have labeled A and B. So we could just add them to the action:

$$\delta A = \int dq \frac{\partial L}{\partial q}(-\dot{q}\epsilon) + \frac{\partial L}{\partial q}\delta\dot{q} + A - B \tag{3.1.18}$$

We do the usual integration by parts:

$$\delta A = \int dt[\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}]\delta q + \delta q \frac{\partial L}{\partial \dot{q}}|_{t_a}^{t_b} + A - B \tag{3.1.19}$$

The term in the brackets is zero as the trajectories obey the E-L equation and $\delta A = 0$ because we have a symmetry. So we get:

$$\delta q \frac{\partial L}{\partial \dot{q}}|_{t_a}^{t_b} + A - B = 0 \tag{3.1.20}$$

The first term shows the same contribution that we saw before from Noether's theorem, that came from integrating by parts. We know that :

$$\delta q = -\epsilon\dot{q} \tag{3.1.21}$$

So we have:

$$-\epsilon\dot{q}\frac{\partial L}{\partial \dot{q}}|_{t_a}^{t_b} + A - B = 0 \tag{3.1.22}$$

Now we need to examine A B; they are both contributions that come from shifting the trajectory by small amount ϵ. So the change in action at A is simply $L(t_a)\epsilon = A$ as the Lagrangian at point A will not have shifted if we take ϵ to be infinitesimal. Same for B; $L(t_b)\epsilon = B$. So now we can put the expressions for these contributions back in equation 3.1.20:

$$\epsilon(-\dot{q}\frac{\partial L}{\partial \dot{q}}|_{t_a}^{t_b} + L(t_b) - L(t_a)) = 0 \tag{3.1.23}$$

Now we can see as we did before that the Lagrangian evaluated between two points along with the first term is zero. So the quantity $L - \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$ is conserved. Which can also be written as:

$$L - \dot{q}_i \pi_i \quad \text{as} \quad \frac{\partial L}{\partial \dot{q}_i} = \pi_i \tag{3.1.24}$$

This is the conserved Noether charge for time translation invariance. This is called the negative of the Hamiltonian, H, or equivalently:

$$H = \dot{q}_i \pi_i - L \tag{3.1.25}$$

which is the total energy of the system. Lets work this quantity for a system to see that this really is the energy; so consider the Lagrangian:

$$L - \frac{1}{2}m\dot{x}^2 - U(x) \tag{3.1.26}$$

The canonical momentum is:

$$\pi = m\dot{x} \tag{3.1.27}$$

So the first term in the Hamiltonian is:

$$\dot{q}_i \pi_i = m\dot{x}^2 \tag{3.1.28}$$

Then subtract the Lagrangian:

$$\begin{aligned} H &= \dot{q}_i \pi_i - L && (3.1.29)\\ &= m\dot{x}^2 - (\frac{1}{2}m\dot{x}^2 - U(x)) && (3.1.30)\\ &= \frac{1}{2}m\dot{x}^2 + U(x) && (3.1.31) \end{aligned}$$

This is infact the definition of energy (a quantity that is conserved under time translation invariance).

3.2 Examples

Now I shall do some examples that show how the new methods of classical mechanics are alot easier to work in, as supposed to Newtonian mechanics.

3.2.1 Simple pendulum

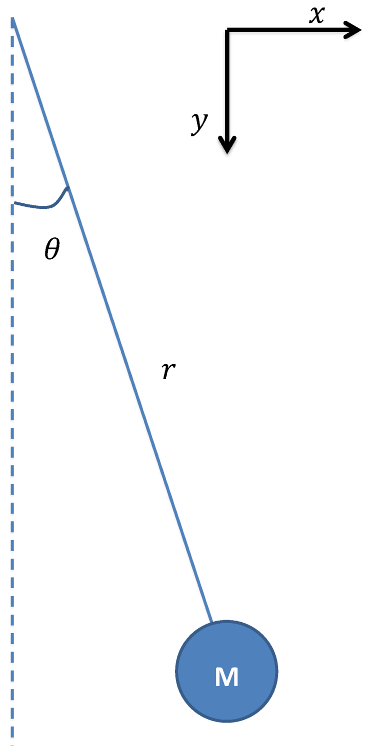


Figure 6: Simple pendulum of rod of length r

In this set-up the only thing that changes with time is θ. Now, the x component of the velocity is:

$$v_x = \dot{\theta} r \cos\theta \tag{3.2.1}$$

and the y component:

$$v_y = -\dot{\theta} r \sin\theta \tag{3.2.2}$$

Therefore the total velocity is simply:

$$\mathbf{v} = (r\dot{\theta}\cos\theta, -r\dot{\theta}\sin\theta) \tag{3.2.3}$$

The kinetic energy is then:

$$T = \frac{1}{2} m r^2 \dot{\theta}^2 (\cos^2\theta + \sin^2\theta) = \frac{1}{2} m r^2 \dot{\theta}^2 \tag{3.2.4}$$

Potential energy:

$$U = -rmg\cos\theta \tag{3.2.5}$$

So the Lagrangian for the system is:

$$L = \frac{1}{2} m r^2 \dot{\theta}^2 + mgr\cos\theta \tag{3.2.6}$$

The canonical momentum for the system is:

$$\pi_\theta = mr^2\dot{\theta} \quad \text{(Angular momentum)} \tag{3.2.7}$$

The E-L equation:

$$\begin{aligned} \frac{d}{dt} mr^2\dot{\theta} &= -mgr\sin\theta \\ r^2\ddot{\theta} &= -gr\sin\theta \end{aligned} \tag{3.2.8}$$

To see that these equations are the same as Newton's laws would give we can compute the Hamiltonian as that gives the total energy of the system:

$$\begin{aligned} H &= \pi_\theta\dot{\theta} - L \\ &= mr^2\dot{\theta}^2 - \left(\frac{1}{2}m^2 + mgr\cos\theta\right) \\ &= mr^2\dot{\theta}^2 - mgr\cos\theta \end{aligned} \tag{3.2.9}$$

This can be immediately recognised as the total energy of the system, with the first term representing the (rotational) kinetic energy and the second is simply the potential energy.

Another way to think about it, is in terms of the moment of inertia,I, which is defined as:

$$I = m\mathbf{r}^2 \tag{3.2.10}$$

where **r** is the perpendicular distance from the pivot of rotation.

Using this we can rewrite the Hamiltonian as:

$$H = \alpha I\dot{\theta}^2 - mgr\cos\theta \tag{3.2.11}$$

Where α is a constant that depends on where the pivot is and the shape and structure of the rod.

3.2.2 Double pendulum

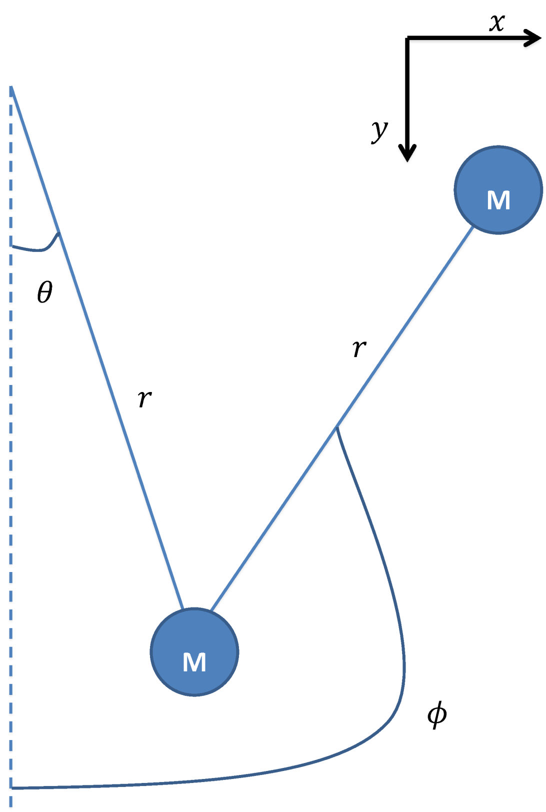


Figure 7: double pendulum of rods of length r each

Lets calculate the kinetic energies first:

$$T = \frac{1}{2}m^2\dot{\theta}^2 \quad \text{ball 1} \tag{3.2.12}$$

Now the second ball will be moving for two reasons, firstly the fact that ball 1 is moving and secondly under its own motion. So we have to add two velocities:

$$v_2 = \left(r\dot{\theta}\cos\theta + r\dot{\phi}\cos\phi, -r\dot{\theta}\sin\theta - r\dot{\phi}\sin\phi\right) \tag{3.2.13}$$

So the kinetic energy is:

$$\frac{1}{2}mv_2^2 = \frac{1}{2}m\left(r^2\dot{\theta}^2 + r^2\theta^2\right) + mr^2\dot{\theta}\dot{\phi}\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right) \tag{3.2.14}$$

The total kinetic energy is then:

$$T = mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mr^2\dot{\theta}\dot{\phi}\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right) \tag{3.2.15}$$

but notice that:

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta - \phi) \tag{3.2.16}$$

Substituting this into the equation above:

$$T = mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + mr^2\dot{\theta}\dot{\phi}\left(\cos(\theta - \phi)\right) \tag{3.2.17}$$

Now the potential energy is:

$$U = -2mgr\cos\theta - mgr\cos\phi \tag{3.2.18}$$

The Lagrangian is then:

$$L = mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + mr^2\dot{\theta}\dot{\phi}\left(\cos(\theta - \phi)\right) + mgr\left(2\cos\theta + \cos\phi\right) \tag{3.2.19}$$

This problem does not have a conserved quantity as there is no symmetry (the rotational symmetry is broken by the gravitational field). Now lets suppose that this is happening in the presence of no gravity. So the Lagrangian is simply the kinetic energy:

$$L = mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + mr^2\dot{\theta}\dot{\phi}\left(\cos(\theta - \phi)\right) \tag{3.2.20}$$

This Lagrangian is now invariant under these transformations:

$$\begin{aligned} \theta &\rightarrow \theta + \epsilon \\ \phi &\rightarrow \phi + \epsilon \end{aligned} \tag{3.2.21}$$

Now the Noether charge for these symmetries can be calculated:

$$Q = \sum_i \pi_i f_i \quad \text{where} \quad \pi_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{as} \quad f_i = 1 \tag{3.2.22}$$

So $\pi_i = \pi_\theta + \pi_\phi$, where:

$$\pi_\theta = \frac{\partial L}{\partial \dot{\theta}} = 2mr^2\dot{\theta} + mr^2\dot{\theta}\left(\cos(\theta - \phi)\right) \tag{3.2.23}$$

$$\pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = 2mr^2\dot{\phi} + mr^2\dot{\theta}\left(\cos(\theta - \phi)\right) \tag{3.2.24}$$

Therefore:

$$\pi_i = 2mr^2\dot{\theta} + mr^2\cos(\theta - \phi)\left(\dot{\phi} + \dot{\theta}\right) \tag{3.2.25}$$

So this is the thing that does not change with time and it is the angular momentum.

Now we can compute the E-L equation to finally obtain the equation of motion for this system[7]:

$$\frac{d}{dt}\left(2mr^2\dot{\theta} + mr^2\dot{\phi}\cos(\theta-\phi)\right) = -mr^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) \tag{3.2.26}$$

3.2.3 Harmonic oscillator

The pendulum is almost a version of the Harmonic oscillator (infact for oscillations near the equilibrium point will be harmonic). The potential energy of the Harmonic oscillator is:

$$U = -mgr + \frac{1}{2}mgr\theta^2 \tag{3.2.27}$$

Infact any function that has a smooth minimum can be approximated as a quadratic term near the minimum. The mgr is a constant in the energy so can be ignored, so the potential energy is just proportional to θ^2 (it is quadratic). As an example lets consider a mass spring system:

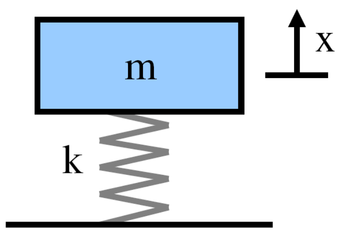


Figure 8: Mass spring system

The potential energy of the system is:

$$U = \frac{kx^2}{2} \tag{3.2.28}$$

So the Lagrangian is:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{kx^2}{2} \tag{3.2.29}$$

Computing the ingredients for the E-L equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} \tag{3.2.30}$$

$$\frac{\partial L}{\partial x} = -kx \tag{3.2.31}$$

[7]solving it is another course in itself!

Therefore the E-L equation is:

$$m\ddot{x} = -kx \tag{3.2.32}$$

This is usually rewritten as:

$$\ddot{x} = -\frac{k}{m}x \tag{3.2.33}$$

This has the solutions of the form:

$$x = A\cos\omega t + B\sin\omega t \quad \text{where} \quad \omega^2 = \frac{k}{m} \tag{3.2.34}$$

Notice that there are two coefficients as the equations are second order. For completeness, the canonical momentum is:

$$p_x = \pi_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{3.2.35}$$

The Hamiltonian is:

$$\begin{aligned} H &= p_x\dot{x} - L \\ &= m\dot{x}^2 - \left(\frac{1}{2}m\dot{x}^2 - \frac{kx^2}{2}\right) \\ &= \frac{1}{2}m\dot{x}^2 + \frac{kx^2}{2} \end{aligned} \tag{3.2.36}$$

Up until now we have dealt with the Lagrangian form of classical mechanics. Next we shall talk about the Hamiltonian formulation of classical mechanics.

4 Hamiltonian formulation

Hamilton decided that he did not want to work with q's $\dot{q}$'s, instead with q's and p's. He same some kind of symmetry between q's and p's, p is generally related to $\dot{x}$:

$$\dot{x} = \frac{p}{m} \tag{4.0.37}$$

So the Hamiltonian for the mass spring system is:

$$H = \frac{p^2}{2m} + \frac{kx^2}{2} \tag{4.0.38}$$

4.1 Hamilton's phase space

In the Hamiltonian formulation, it is useful to look at the phase space. For this particular system the phase space looks like:

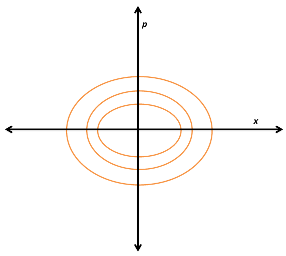


Figure 9: Phase space of Hamiltonian 4.0.38, with trajectories of the system with different total energies

Points in phase space represent the state of the system. To track its trajectory through phase space, we can look at the Hamiltonian (remember the Hamiltonian is just the energy, and the energy of a closed is always constant). We see that the equation has the form that describes an ellipse, in fact if the coefficients of p^2 and x^2 are the same then it would be the equation of a circle (the total energy determines the radius).

Changing the energy will change the orbit by changing the radius. The trajectories intercept the axis when $p = 0$ or $x = 0$. When $p = 0$:

$$x = \sqrt{\frac{2E}{k}} \tag{4.1.1}$$

When $x = 0$:

$$p = \sqrt{2mE} \tag{4.1.2}$$

The time it takes for the system to complete a full rotation in phase space depends on the angular velocity ω. The longer ω the shorter the time it takes for it to go around.

Suppose you project the motion on to the x-axis; it would just move back and forth, it oscillates! And the same for the p-axis. All systems move on surfaces of constant energy (they do not have to be elliptical or circular). Suppose you now take a little patch of area in phase space, this patch represents the uncertainty in the knowledge of the initial conditions, it also just moves in phase space as any other point, and the important thing is that it preserves its area over time.

In fact, the whole point of Hamiltonian formalism of mechanics is that no points in phase space are lost (another way of saying the area/volume is conserved). Newton's equations are not of this type; suppose you are at a point in phase space, one cannot predict the system's future without knowing another piece of information, such as the velocity. The equation itself is a second order DE:

$$m\frac{d^2x_i}{dt^2} = F_i \tag{4.1.3}$$

Suppose there are N such DE's, it is easy to see that these can equivalently be written as 2N first order DE's. First lets define, something that we already knwo:

$$m\frac{d_{x_i}}{dt} = p_i \tag{4.1.4}$$

So:

$$m\frac{d^2x_i}{dt^2} = \frac{\partial p_i}{dt} = F_i \tag{4.1.5}$$

Now we will have twice as many equations which involve x_i and p_i coordinates.

Since these are first order DE. Now one might think that we only need one piece of information to predict the evolution of the system, however, there are two equations, so two pieces of information would still be required (this just shows that there is no new physics coming out of this formalism, it is just (yet) another way of rewriting Newton's equations.

4.2 Legendre transformations & Hamilton's equations

Firstly consider the simplest possible case; we have two variables v and p, but these are not really two variables as they are functions of each other ($p = mv$). Now suppose they are *single valued functions* of each other (i.e one v only corresponds to one p). So the graph of p and v looks something like:

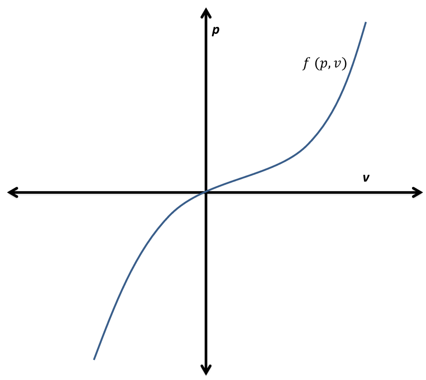


Figure 10: Single valued function of p and v, $f(p, v)$

Or more specifically, it *cannot* look like:

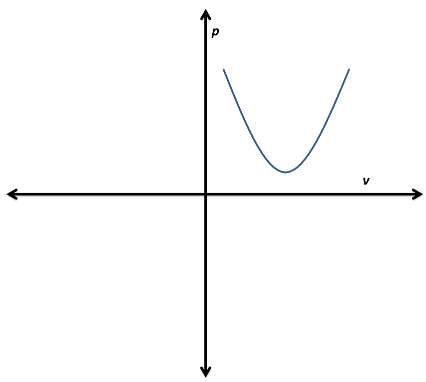


Figure 11: Double valued function of p and v

as a single value of p would have 2 values of v. For single valued functions, it is possible to invent a pair of functions like:

$$L(v) \quad \text{such that} \quad \frac{\partial L(v)}{\partial v} = p \tag{4.2.1}$$

$$H(p) \quad \text{such that} \quad \frac{\partial H(p)}{\partial p} = v \tag{4.2.2}$$

To see why we can write these functions, lets take the function drawn in **??** and solve the equation for L:

$$L(v) = \int_0^v p dv \tag{4.2.3}$$

So we take the integral from zero. On the graph we saw before that L is just the area under the curve.
The same can be done for H:

$$H(p) = \int_0^p v dp \tag{4.2.4}$$

Therefore H+L is the area of the rectangle which is given by $p \times v$, or it can also be written as:

$$H = pv - L \tag{4.2.5}$$

This sets up a clear method, start with the Lagrangian, $L(v)$, then differentiate w.r.t v to get p and then use:

$$H(p) = pv(p) - L(v(p)) \tag{4.2.6}$$

Now consider a small change of p and see how H changes:

$$\delta H = p\delta v + v\delta p - p\delta v = v\delta p \tag{4.2.7}$$

So:

$$\frac{dH}{dp} = v \tag{4.2.8}$$

Which is how we defined the Legendre transformation. Now we ask what happens if there are many v's, then the definitions become:

$$\frac{\partial L}{\partial v_i} = p_i \tag{4.2.9}$$

$$H(p_i) = \sum_i p_i v_i - L(v_i) \tag{4.2.10}$$

$$\frac{\partial H}{\partial p_i} = v_i \tag{4.2.11}$$

Even though we have used pv as variables (which of course are just the momentum and the velocity respectively), these definitions are completely general and work for any variables that are related by a *single valued* function. This was just a piece of mathematics.

In mechanics, the Lagrangian also depends on positions as well as velocities.

In terms of the Legendre transformations, having the positions in there would not have made any difference as we never differentiate w.r.t the position. So the mechanics description of the relation between the Lagrangian and the Hamiltonian is:

$$H = \sum_i p_i v_i - L \tag{4.2.12}$$

Now lets see how the Hamiltonian changes with small changes in both p's and q's:

$$\delta H = p_i \delta v_i + v_i \delta p_i - \frac{\partial L}{\partial q_i} \delta q_i - \frac{\partial L}{\partial v_i} \delta v_i \tag{4.2.13}$$

But we know that:

$$\frac{\partial L}{\partial v_i} = p_i \tag{4.2.14}$$

So the final equations simplify to:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{4.2.15}$$

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \tag{4.2.16}$$

Now recall the E-L equation, it basicly states:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{4.2.17}$$

Substitute this into Eq 4.2.16:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{4.2.18}$$

This completes the derivation of the two equations, that are called *Hamilton's equations*. I will write them out again (I would draw a box around them if I knew how to do that, they are that important!):

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{4.2.19}$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{4.2.20}$$

Here we see that mechanics gets re-packaged as a system of (two) first order DE of great simplicity. All we need is one function of p's and q's and then we immediately know how the point moves through phase space. (If you know p and q, we can get $\dot{q}$ and $\dot{p}$ using Hamilton's equations. One of the surprising things this shows, is that with very different physical physical interpretation of p's and q's there is a very profound symmetry between them. Its the study of these equations that defines modern mechanics.

At every point in phase space one can calculate $\frac{\partial H}{\partial q_i}$ and $\frac{\partial H}{\partial p_i}$ to get $\dot{q}_i$ and $\dot{p}_i$ respectively. This defines a kind off velocity through phase space. Notice that its the derivatives of the Hamiltonian w.r.t q and p as supposed to q and $\dot{q}$'s and we shall see that this has a profound implication.

4.3 1D particle and energy conservation

Starting with the simplest system, the Lagrangian is the same as before:

$$L = \frac{1}{2}mv^2 - U(x) \tag{4.3.1}$$

The Hamiltonian is:

$$H = \frac{p^2}{2m} + U(x) \tag{4.3.2}$$

Now compute Hamilton's equations:

$$\frac{\partial H}{\partial p} = \dot{x} \tag{4.3.3}$$

$$-\frac{\partial H}{\partial x} = \dot{p} \tag{4.3.4}$$

Which is just equivalent to Newton's second law. We can also test for energy conservation by differentiating the Hamiltonian w.r.t time. If it doesn't change then the energy is conserved. So lets compute it:

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i \tag{4.3.5}$$

But we know that:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

Substitute this into 4.3.5:

$$\frac{\partial H}{\partial T} = -\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} = 0 \tag{4.3.6}$$

Which shows that energy is indeed conserved. This can be thought of in terms of phase space; a system represented in phase space will move on contours of constant energy in phase space. They can be closed curves aswell as lines (like we previously studied in the Harmonic oscillator)

4.4 Poisson brackets

We know from the Lagrangian formalism that conservation laws have to do with symmetries. Lets ask weather a quantity, A, is conserved or not. Suppose A is a function of q_i's and p_i's, so that at every point in phase space A has a *unique* value. We want to know weather the time derivative of A is zero:

$$\begin{aligned} \frac{d}{dt}A(p_i, q_i) &= \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial q_i}\dot{q}_i \\ \dot{A} &= -\frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} \end{aligned} \tag{4.4.1}$$

So we find that the time derivative of any quantity is a sum of two terms. To generalise, we have to introduce a new notation of Poisson brackets. So for any two functions; $A(q_i, p_i)$ and $B(q_i, p_i)$, the Poisson bracket is defined as:

$$\{A(p_i, q_i), B(p_i, q_i)\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{4.4.2}$$

The Poisson bracket of any quantity with the Hamiltonian is its time derivative. This is yet another formalism of classical mechanics and its a generalisation of Hamilton's equations. As an example $\dot{q}$ must be the Poisson bracket of q with H:

$$\dot{q} = \{q, H\} \tag{4.4.3}$$

To see this, let us compute the Poisson bracket:

$$\frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial H}{\partial q} = \{q, H\} \tag{4.4.4}$$

But we know that:

$$\frac{\partial q}{\partial q} = 1 \quad \text{and} \quad \frac{\partial q}{\partial p} = 0$$

Using these along with the equation above it gives:

$$\dot{q} = \frac{\partial H}{\partial p} \tag{4.4.5}$$

Which is what we expected from Hamilton's equations. Similarly we can show that:

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{4.4.6}$$

Here we see that both of Hamilton's equations are special cases of a much more general rule that the time derivative of a function is the Poisson bracket of the function with the Hamiltonian (striking similarity with Ehrenfest's theorem in Quantum mechanics).

4.5 Liouville's Theorem

This theorem is at the heart of the Hamiltonian formalism of classical mechanics. It can be thought of as a generalisation of the idea of conservation of momentum. It is easiest to explain using diagrams of phase space, so consider:

Figure 12: Phase space showing points of a system

Suppose the phase space is populated with a uniform density of points. The points combined together will occupy some volume (I an saying volume, but it could be a space in many dimensions depending on how many dynamical variables are in the system):

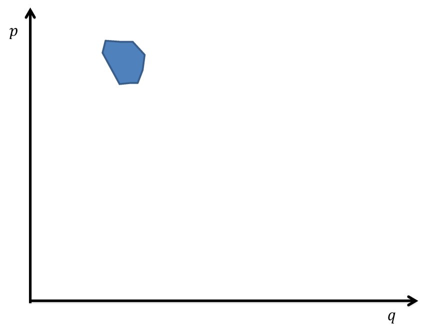


Figure 13: Phase space showing volume patch occupied by points

As a system evolves the points in phase space will move around, so the shape of the patch they cover will also change. However according to Liouville's theorem the volume of patch in phase space always stays the same and the shape always maintains its topology. What is not generally conserved is the distance between the points. This would still conserve the volume as the shape can stretch in one direction and compress in another direction.

The beauty of the Hamiltonian form of mechanics is this flow of volume in phase space, as the entire flow is determined by the Hamiltonian. According to Liouville's theorem the flow is incompressible, to see what this means lets consider a flow in 1D:

Figure 14: Points in 1D phase space

The density of the points is the same everywhere on the line. There is only one possible motion that corresponds to an incompressible flow in this case and that is the points simply move together with the same velocity along the line. If the velocity was different, say faster at the back then at the front, then the points would clump together.

Another way to say this would be to consider any given region on the line and observe it over time. If the flow is incompressible, then the total number of points leaving that space will be equal to the total number of points entering that space in any given time interval.

So we can write that the change in velocity must be zero:

$$\frac{\partial v}{\partial x}\delta x = 0 \tag{4.5.1}$$

δx is finite but infinitesimal therefore $\frac{\partial v}{\partial x} = 0$ for an incompressible fluid which is what we would intuitively imagine. Now lets move to 2D:

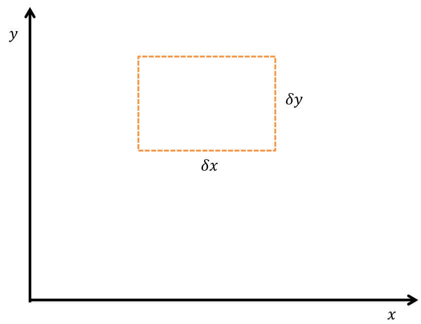


Figure 15: Infinitesimal patch in phase space

We want to see how the number of points in the square above changes over time (again we are implicitly assuming that the density is uniform). The number of particles coming into the square through the vertical edges is given by following the same logic as we used for the 1D case:

$$\frac{\partial V_x}{\partial x}\delta y \delta x \propto N \tag{4.5.2}$$

where N is the number of particles. For the horizontal edges:

$$\frac{\partial V_y}{\partial y}\delta x\delta y \propto N \tag{4.5.3}$$

Therefore the net flux of particles is given by:

$$\begin{aligned} N &\propto \frac{\partial V_x}{\partial x}\delta x\delta y + \frac{\partial V_y}{\partial y}\delta x\delta y \\ &\propto \delta x\delta y\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) \end{aligned} \tag{4.5.4}$$

So the flux which is the total number of particles passing a unit area is proportional to the *divergence* of the velocity vector:

$$\frac{N}{\delta x\delta y} \propto \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) \tag{4.5.5}$$

If the fluid is incompressible then the net flux must be zero, so the divergence is also zero:

$$\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) = div(v) = 0 \tag{4.5.6}$$

Now lets come back to phase space. The x's are just coordinates so in the phase space they represent the p's and q's. So the number of x's is twice the number of q's (as we also need an equal number of q's). The $\dot{q}$'s and $\dot{p}$'s are the local velocities in the phase space. Lets calculate the divergence of the flow:

$$\frac{d\dot{p}_i}{dp_i} = \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) \tag{4.5.7}$$

$$\frac{d\dot{q}_i}{dq_i} = \frac{\partial}{\partial q_i}\left(-\frac{\partial H}{\partial p_i}\right) \tag{4.5.8}$$

$$divv_p s = \frac{d\dot{p}_i}{dp_i} + \frac{d\dot{q}_i}{dq_i} \tag{4.5.9}$$

where $v_p s$ is the velocity in phase space. Combining these equations gives:

$$\frac{d\dot{p}_i}{dp_i} + \frac{d\dot{q}_i}{dq_i} = \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial}{\partial q_i}\left(-\frac{\partial H}{\partial p_i}\right) = 0 \tag{4.5.10}$$

So we see that the divergence is zero.

Consider a particle in free motion (no potentials), and lets make up a Lagrangian:

$$L = \frac{\dot{x}^2}{2} \tag{4.5.11}$$

The canonical momentum is:

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} \tag{4.5.12}$$

Suppose we define a new coordinate:

$$y = \alpha x \tag{4.5.13}$$

where α is a constant. The derivative is:

$$\dot{x} = \frac{\dot{y}}{\alpha} \tag{4.5.14}$$

Rewriting the Lagrangian in terms of y:

$$L = \frac{\dot{y}^2}{2\alpha^2} \tag{4.5.15}$$

The canonical momentum is:

$$\frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\alpha^2} = p_y \tag{4.5.16}$$

Which is the same as:

$$p_y = \frac{\dot{x}}{\alpha} = \frac{p_x}{\alpha} \tag{4.5.17}$$

So we see:

$$p_y = \frac{1}{\alpha} p_x \tag{4.5.18}$$

Notice what happens when we make a coordinate transformation that stretches the x-axis, it shrinks the p-axis.

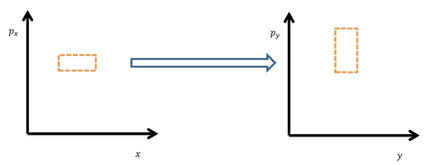


Figure 16: Coordinate transformation from x to y

4.6 Chaotic systems

Chaotic systems also have the same property that the area in phase space is conserved. The idea behind chaotic systems is that we cannot know the value of the p's and q's with infinite precision, such that they are represented by points in phase space. In reality they will have some uncertainty in those values (usually depending on the apparatus being used to observe them).

Instead what we get is small spheres (or rectangles) as supposed to point, in phase space that can be observed. The important thing to realise is that even though the best one can observe is a sphere of a specific dimension, the real particles in the world will have *precise* values (remember we are ignoring QM here). Therefore there could be very many (infact, an infinite number if it was not for QM) points within our smallest possible viewable sphere, which over time we cannot keep track off.

The next thing that happens in chaotic systems is that they will form very large and spread out (fractalated) structures in phase space over time. They still obey Liouville's theorem, however due to the limitations of our ability to measure things the volume they occupy in phase space appears to be larger.

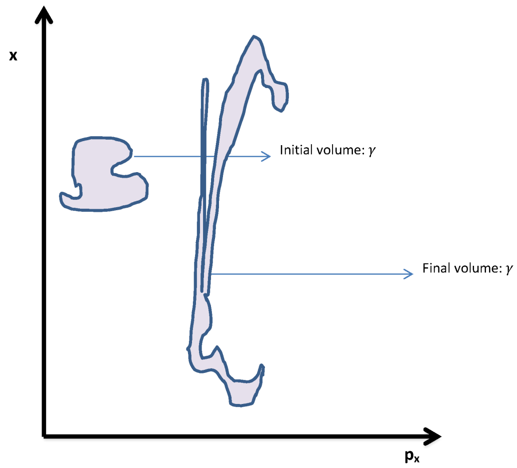


Figure 17: Chaotic system

Most systems in nature are chaotic (given enough time and this is the origin of the second law of thermodynamics)

4.7 Electromagnetic field

The electromagnetic(E-M) field was put together by Maxwell in the 19^{th} century, before this electricity and magnetism were seen as two different phenomena. The electric and magnetic fields are both vector fields. The E-M field has one major difference from the systems that we have studied so far because the magnetic force depends on the velocity of the particle.

4.7.1 Magnetic field, B

The magnetic field is usually denoted by **B**. The force on a particle moving in a magnetic field is:

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \tag{4.7.1}$$

We need one other concept, which is the vector potential, **A**. This is because, one cannot write the mechanics of a charged particle just in terms of the magnetic field. The vector potential is defined by the condition that the magnetic field is the curl of the vector potential:

$$\mathbf{B} = curl(\mathbf{A}) = \nabla \times \mathbf{A} \tag{4.7.2}$$

The special thing about things that are a curl of something is that they do not have a divergence. This follows from the vector calculus identity:

$$div\,(curl(\mathbf{A})) = 0 \tag{4.7.3}$$

Where $\mathbf{A}$ is any vector. So taking the divergence of both sides of Eq 4.7.1:

$$div(\mathbf{B}) = div\,(curl(\mathbf{A})) = 0 \tag{4.7.4}$$

But we shall assume that the vector potential is a fundamental object and that $\mathbf{B}$ comes from it. Now the force on a charged particle can be written as:

$$\mathbf{F} = q\mathbf{v} \times (\nabla \times \mathbf{A}) \tag{4.7.5}$$

Suppose we want the z component of force:

$$\mathbf{F}_z = q\left(\mathbf{v}_x\mathbf{B}_y - \mathbf{v}_y\mathbf{B}_x\right) \tag{4.7.6}$$

In terms of $\mathbf{A}$:

$$\mathbf{F}_z = q\{\mathbf{v}_x\left(\partial_z\mathbf{A}_x - \partial_x\mathbf{A}_z\right) - \mathbf{v}_y\left(\partial_y\mathbf{A}_z - \partial_z\mathbf{A}_y\right)\} \tag{4.7.7}$$

Now it is not obvious how this leads to Hamilton or Lagrange's equations, however we can guess an action and see what it gives:

$$\begin{aligned} S &= \int L dt \\ &= \int \frac{mv^2}{2} dt + q \int \mathbf{A} \cdot dx_i \end{aligned} \tag{4.7.8}$$

The fist term is obvious as it is just the kinetic energy, but the second term is a guess at the simplest possible term that relates the charge and the vector potential. It is simply showing that suppose there is a vector potential ,A, at each point in space, we simply integrate over small intervals in space. In other words, take the component of the magnetic field along a given direction and add up over all of the path.

The action can be rewritten as:

$$S = \int \frac{m\dot{x}^2}{2} dt + q \int \mathbf{A} \cdot \dot{x} dt \tag{4.7.9}$$

So the Lagrangian is:

$$L = \frac{m\dot{x}^2}{2} + q\mathbf{A} \cdot \dot{x} \tag{4.7.10}$$

Now we want to see that the Lagrangian equations lead to the equation for force that we previously saw. Firstly, the canonical momentum:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x \tag{4.7.11}$$

And similar equations for the y and z components. Sometimes $m\dot{x}$ is called the mechanical momentum and the whole term is called the canonical momentum.

Now lets work out the EOM:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z} \tag{4.7.12}$$

and similar equations for x and y. Computing the components:

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{z}} &= m\ddot{z} + q\dot{\mathbf{A}}_z \\ &= m\ddot{z} + q\left(\frac{\partial A_z}{\partial x}\dot{x} + \frac{\partial A_z}{\partial y}\dot{y} + \frac{\partial A_z}{\partial z}\dot{z}\right) \end{aligned} \tag{4.7.13}$$

$$\frac{\partial L}{\partial z} = q\left(\dot{x}\frac{\partial A_x}{\partial z} + \dot{y}\frac{\partial A_y}{\partial z} + \dot{z}\frac{\partial A_z}{\partial z}\right) \tag{4.7.14}$$

So the E-L equation is:

$$m\ddot{z} + q\left(\frac{\partial A_z}{\partial x}\dot{x} + \frac{\partial A_z}{\partial y}\dot{y} + \frac{\partial A_z}{\partial z}\dot{z}\right) = q\left(\dot{x}\frac{\partial A_x}{\partial z} + \dot{y}\frac{\partial A_y}{\partial z} + \dot{z}\frac{\partial A_z}{\partial z}\right) \tag{4.7.15}$$

Moving the second term on the L.H.S to the R.H.S we get:

$$\begin{aligned} m\ddot{z} &= q\left(\dot{x}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \dot{y}\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right)\right) \\ &= q\left(\dot{x}B_y - \dot{y}B_x\right) \end{aligned} \tag{4.7.16}$$

Which returns the equation we had before:

$$\mathbf{F} = q\left(\mathbf{v} \times \mathbf{B}\right) \tag{4.7.17}$$

As expected we have found the Lagrangian formulation for particles moving in a **B** field. Notice that even though the Lagrangian depends on the vector potential , the equations of motion do not (they only depend on **B**). So we can vary the vector potential in certain ways that does not change the EOM and this will be called choosing a gauge and will be discussed later.

Looking at this equation for the force, one might know parallels of the **B** force to the frictional force in nature due to the velocity dependence. But there is a fundamental difference between them. The frictional force acts along the axis of the motion of a particle. On the other hand the force due to the **B** field acts perpendicular to **v** and **B** due to the cross product.

Now lets calculate the Hamiltonian:

$$
\begin{aligned}
H &= \mathbf{p}_i \dot{x}_i - L = \\
m\dot{x}_i^2 + qA_{x_i}\dot{x}_i - \left(\frac{m\dot{x}_i^2}{2} + q\dot{x}_i A_i\right) &= \frac{m\dot{x}^2}{2}
\end{aligned}
\tag{4.7.18}
$$

Which is just the kinetic energy. This shows that the **B** field does not contribute to the energy, when expressed in terms of velocities. This is equivalent to saying the **B** fields cannot do work. In this form we cannot use Hamilton's equations as they require momenta as supposed to velocities. So lets rewrite $\dot{x}_i$ in terms of the momenta:

$$\dot{x}_i = \frac{p_i - qA_i}{m} \tag{4.7.19}$$

So the kinetic energy (Hamiltonian) is:

$$H = \frac{(p_i - qA - i)^2}{2m^2} \tag{4.7.20}$$

So we see that all of a sudden when expressed in terms of the canonical momenta, the expression for the energy does change and we can now apply Hamilton's equations (which would give the same EOM).

4.7.2 2D particle

Now lets consider the motion of a non-relativistic particle under the influence of an E-M field in 2D. The force is:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4.7.21}$$

This is known as the Lorentz force. Recall that:

$$\mathbf{E} = -\nabla V \tag{4.7.22}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.7.23}$$

where V is the potential:

$$U = qV \tag{4.7.24}$$

The only action that leads to the EOM of the Lorentz force are written in terms of the vector potential A:

$$S = \int \left(\frac{1}{2}m\dot{x}_i^2 - qV\right) dt + \int q\mathbf{A}_i \cdot dx_i \tag{4.7.25}$$

Lets rewrite this as:

$$S = \int \frac{1}{2}m\dot{x}_i^2 dt + q\int \mathbf{A}_i dx_i - V dt \tag{4.7.26}$$

The equation in this form shows a striking symmetry in the potential terms. Its almost as if $\mathbf{A}$ is related to space in the same way that V is related to time. Infact if written as four vectors in terms of the special theory of relativity we could simply write a single potential that was a four vector for the E- field:

$$\int \mathbf{A}_i dx_i - V dt = \int \mathbf{A}_\mu dx^4 \qquad (4.7.27)$$

The notation here is $A_m u = A_{0,1,2,3}$ where $A_3 = V$.

The second term can be rewritten as:

$$q \int dt \left(\mathbf{A} \cdot \mathbf{v} - V\right) \qquad (4.7.28)$$

So the Lagrangian becomes:

$$L = \frac{1}{2} m \dot{x}_i^2 + q \mathbf{A} \dot{\mathbf{v}} - V \qquad (4.7.29)$$

Now lets examine what we can do to the vector potential that does not change the $\mathbf{B}$ field or the equations of motion. This can be thought of as a symmetry as changing the vector potential has to influence on the motion, its called gauge invariance. First lets examine the z component:

$$(\nabla \times \mathbf{A})_z = \partial_x \mathbf{A}_y - \partial_y \mathbf{A}_x \qquad (4.7.30)$$

Suppose I add a gradient to the vector potential:

$$\mathbf{A} \to \mathbf{A} + \frac{\partial}{\partial x_i} \lambda(x, y) \qquad (4.7.31)$$

And similar equations for the y component.

Lets recalculate the $curl A$ now:

$$\nabla \times \mathbf{A} = \partial_x \mathbf{A}_y - \partial_y \mathbf{A}_x + \partial_x \partial_y \lambda - \partial_y \partial_x \lambda \qquad (4.7.32)$$

But the last two terms are the same and therefore cancel, leaving the $curl(A)$ unchanged. In summary, adding the gradient of a scalar to a vector does not change its curl. This can also be seen from the trigonometric identity:

$$curl\left(grad(\lambda)\right) = 0 \qquad (4.7.33)$$

for any scalar λ. Now if we calculate the canonical momentum for this Lagrangian we get:

$$p_x = m\dot{x} + q\mathbf{A}_x \qquad (4.7.34)$$

But now we see that if $\mathbf{A}$ is changed the momentum also changes which means the momentum is not gauge invariant. Now lets compute the E-L equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = m\ddot{x} + q \frac{\partial A_x}{\partial x} \dot{x} = q \frac{\partial A_x}{\partial y} \dot{y} \qquad (4.7.35)$$

$$\frac{\partial L}{\partial x} = q\frac{\partial A_x}{\partial x}\dot{x} + q\frac{\partial A_y}{\partial y}\dot{y} - q\frac{\partial V}{\partial x} \tag{4.7.36}$$

So the E-L equation is:

$$m\ddot{x} + q\frac{\partial A_x}{\partial x}\dot{x} = q\frac{\partial A_x}{\partial y}\dot{y} = q\frac{\partial A_x}{\partial x}\dot{x} + q\frac{\partial A_y}{\partial y}\dot{y} - q\frac{\partial V}{\partial x} \tag{4.7.37}$$

Which simplifies to:

$$m\ddot{x} = -q\frac{\partial V}{\partial x} + q\dot{y}B_z \tag{4.7.38}$$

This reproduces the x-component of the Lorentz force. So even though the momentum was not gauge invariant, the EOM is gauge invariant. Whenever one hears about gauge invariance, we are always talking about some kind of redundancy that is necessary to work out the canonical formulation of the theory.

Now we can consider an example of a uniform magnetic field, this is the simplest situation. Suppose we want B_z to be independent of position:

$$B_z = \partial_x A_y - \partial_y A_x \tag{4.7.39}$$

Lets say the magnitude of the magnetic field is:

$$|B| = b \tag{4.7.40}$$

Suppose we choose the following gauge:

$$\begin{aligned} A_y &= bx \\ A_x &= 0 \end{aligned} \tag{4.7.41}$$

In this case it is clear that B_z is constant, substitute Eq 4.7.41 into Eq 4.7.39 :

$$\begin{aligned} \partial_x A_y &= b \\ {}_y A_x &= 0 \end{aligned} \tag{4.7.42}$$

Which proves Eq 4.7.39. So this is one possible choice of vector potential which will give rise to a uniform magnetic field. Other possibilities include:

$$\begin{aligned} A_y &= 0 \\ A_x &= -bx \end{aligned} \tag{4.7.43}$$

$$\begin{aligned} A_y &= \frac{1}{2}bx \\ A_x &= -\frac{1}{2}bx \end{aligned} \tag{4.7.44}$$

These different vector potentials are related by gauge transformations. Infact, there is some scalar that can take us from one to the other set of potentials. We can choose any of these gauge's as the equations of motion do not depend on A, they just depend on B

Firstly, lets focus of the first gauge. The Lagrangian is:

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + qA_x\dot{x} \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + qby\dot{x} \end{aligned} \tag{4.7.45}$$

Now lets calculate the canonical momenta conjugate to x and y:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qby \tag{4.7.46}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \tag{4.7.47}$$

Remember that momentum conservation comes about due to invariance under translation in position. So we can immediately see that the p_x is a conserved quantity as there is no x dependence in the Lagrangian. In other words the equations of motion are unchanged under the transformation:

$$x \to \ +\epsilon \tag{4.7.48}$$

But under the transformation:

$$y \to y + \epsilon \tag{4.7.49}$$

the equations of motion do change due to the y dependence in the Lagrangian. If p_x is initially zero, it will always be zero, so we get:

$$\begin{aligned} m\dot{x} + qby &= 0 \\ \dot{x} &= \frac{qb}{m}y \end{aligned} \tag{4.7.50}$$

But this y dependence came about from the choice of gauge we made. If we choose the second gauge shown above, the Lagrangian becomes:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - qbx\dot{y} \tag{4.7.51}$$

It is clear that following the same procedure as we did for the previous gauge, this will conserve P_y. So for this gauge we get:

$$P_y = m\dot{y} - qbx \tag{4.7.52}$$

Hence it is seen that:

$$\begin{aligned}\dot{y} &\propto x \\ \dot{x} &\propto y\end{aligned}$$

These equations actually describe circular motion, as they have solutions of the form:

$$x = r\cos\omega t \tag{4.7.53}$$
$$y = r\sin\omega t \tag{4.7.54}$$

where[8]:

$$\omega = \frac{qb}{m} \tag{4.7.55}$$

Here we have reformulated the very basic calculations of E-M in terms of the new formulation of mechanics. Since we assumed that the magnetic field was completely homogenous, the same results should be obtained in any region of space. To see this, we can simply shift the point about which the charged particle is moving as follows:

$$\begin{aligned}x &= r\cos\omega t + x_0 \\ y &= r\sin\omega t + y_0\end{aligned}$$

This will not change $\dot{x}$ and $\dot{y}$. Now lets recalculate the conjugate momenta to x:

$$p_x = m\dot{x} + qby = -mr\omega\sin\omega t + qbr\sin\omega t + y_0 \tag{4.7.56}$$

Now if we substitute for ω we see that things start to cancel and what is left is:

$$p_x = y_0 bq \tag{4.7.57}$$

And a similar expression is obtained for p_y:

$$p_y = -qbx_0 \tag{4.7.58}$$

Here we see a remarkable result that the momenta are simply proportional to the position of the point around which the particle is rotating. So another way to think about this is that the conservation tells us that the circle in which the particle is moving does not move itself (as the momentum depends on where it is).

Now lets put the electric field back in. For now lets say the electric field, **E** is just in the x-direction:

$$V = -Ex \tag{4.7.59}$$
$$\frac{\partial V}{\partial x} = -E \tag{4.7.60}$$

[8] Recall this as the cyclotron frequency of charged particle in magnetic field

So the Lagrangian in the first gauge becomes:

$$L = \frac{1}{2}m(\dot{x}^2\dot{y}^2) - qbx\dot{y} + qEx \tag{4.7.61}$$

The canonical momentum is still the same. But it is no longer conserved; as there is an explicit x dependence in the Lagrangian. We can compute the E-L euqation to see this more clearly:

$$\frac{d}{dt}p_x = \frac{\partial L}{\partial x} = qE \tag{4.7.62}$$

So we see that this just gives back the equation for force on charged particle in an electric field:

$$F_x = qE_x$$

But the p_y remains unchanged as the electric potential does not depend on y. Therefore the new equations of motion are:

$$m\dot{x} + qbdoty \quad = \quad qEm\dot{y} - qb\dot{x} \qquad = 0(4.7.63)$$

Lets solve these equations. For simplicity lets look for solutions with no acceleration, so $m\ddot{x}$ and $m\ddot{y}$ are zero. Hence:

$$\dot{y} = \frac{E}{b} \tag{4.7.64}$$

$$\dot{x} = 0 \tag{4.7.65}$$

We see that $\dot{y}$ is constant as we expected and so is $\dot{x}$. So in the $x - y$ plane the motion is like:

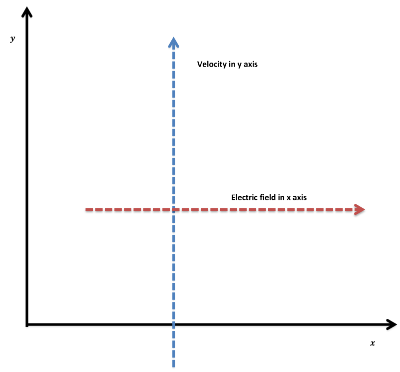


Figure 18: Motion of charged particle in **E** field

The interesting thing here is that we put the force in the x direction and it causes the particle to move in the y direction, which is completely counter intuitive. Its like pushing an object sideways and starts moving vertically!! This remarkable phenomena is called the *Hall effect*.

5 Poisson brackets & canonical transformations

We return to the description of Poisson brackets as it is a whole different description of mechanics and this time we will see some of the applications of it. This is a highly abstract form of mechanics (infact there would be no point in learning it, if it were not so closely related to QM).

5.1 Review

Lets begin with Hamilton's equations:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{5.1.1}$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{5.1.2}$$

Lets take some function of p_i and q_i, $A(p_i, q_i)$. As the function moves through phase space, its value changes over time. This is not because the function changes over time, but because the q_i's and p_i's change. To see how $A(p_i, q_i)$ changes along its trajectory through phase space, we can differentiate it:

$$\dot{A} = \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} \tag{5.1.3}$$

For a general pair of functions, $A(p_i, q_i)$ and $B(p_i, q_i)$ a Poisson bracket is defined as:

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{5.1.4}$$

Therefore combining this with the equation above:

$$\frac{d}{dt}A(p_i, q_i) = \{A, H\} \tag{5.1.5}$$

If $A(p_i, q_i) = p$, we get:

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{5.1.6}$$

This is just one of Hamilton's equations. Now if $A(p_i, q_i) = p$ we get the second Hamilton equations:

$$\frac{dp}{dt} = \frac{\partial H}{\partial p} \tag{5.1.7}$$

5.2 Properties of Poisson brackets

Here I shall list a set of properties that the Poisson brackets have, they can simply be thought of as definitions (I will not prove all of them, but they are quite straightforward to derive):

$$\{A, B\} = -\{B, A\} \quad \text{Anti-commutation} \tag{5.2.1}$$

$$\{p_i, p_j\} = \{q_i, q_j\} = 0 \tag{5.2.2}$$

$$\{q_i, p_j\} = \delta_{ij} \tag{5.2.3}$$

$$\{p_j, q_i\} = -\delta_{ji} \tag{5.2.4}$$

$$\{p_i, F(q_i, p_i)\} = -\frac{\partial F}{\partial q_i} \tag{5.2.5}$$

$$\{p_i, F(q_i, p_i)\} = -\frac{\partial F}{\partial q_i} \tag{5.2.6}$$

$$\{q_i, F(q_i, p_i)\} = \frac{\partial F}{\partial p_i} \tag{5.2.7}$$

$$\{\alpha A, B\} = \alpha\{A, B\} \quad \text{Linearty} \tag{5.2.8}$$

$$\{A + C, B\} = \{A, B\} + \{C, B\} \tag{5.2.9}$$

$$\begin{aligned}
\{AB, C\} &= \frac{\partial(AB)}{\partial q}\frac{\partial C}{\partial p} - \frac{\partial(AB)}{\partial p}\frac{\partial C}{\partial q} \\
&= A\frac{\partial B}{\partial q}\frac{\partial C}{\partial p} - A\frac{\partial B}{\partial p}\frac{\partial C}{\partial q} + B\frac{\partial A}{\partial q}\frac{\partial C}{\partial p} - B\frac{\partial A}{\partial p}\frac{\partial C}{\partial q} \\
&= A\{B, C\} + B\{A, C\}
\end{aligned} \tag{5.2.10}$$

5.3 Canonical transformations

Phase space has a structure to it. Structures are things that are invariant under various transformations. What Poisson brackets do; is they describe *flows* in phase space (like the motion of a fluid represents a flow). One kind of flow is just the motion of the phase space (or the points in it) with time. For example we can consider flows in coordinate space (i.e x-y axis). We can think of a rotation in space as defining a kind of flow in space. To do this imagine the rotation taking place over an infinite number of intermediate steps (i.e a continuous transformation).

Transformations are changes in a system that do not change the dynamics of a system, i.e the system has some forms of symmetries. In the cases that have been looked at so far, the symmetries were typically of the form:

$$Q = Q(q) \tag{5.3.1}$$

So the variable q or p is changed without mixing it with the other (i.e a momentum never becomes a position and vice-verse). The question now is; are there more interesting symmetries in nature that do mix up the p's and the q's, without changing the physics of the system.

We want to find these transformations. Suppose we have a phase space with p's and q's and we make the following transformations:

$$\begin{aligned} P &= 2p \\ Q &= 2q \end{aligned} \tag{5.3.2}$$

Now we have to check weather this preserves the Poisson bracket structure (as this is what describes the dynamics of the system). So lets calculate the Poisson bracket:

$$\{P, Q\} = 4 \tag{5.3.3}$$

which is clearly not the same as Eq 5.2.3, so the Poisson bracket properties are not satisfied and this is not a valid transformation. Now lets try the transformation:

$$\begin{aligned} P &= \frac{1}{2}p \\ Q &= 2q \end{aligned} \tag{5.3.4}$$

Now the Poisson bracket is:

$$\{P, Q\} = 1 \tag{5.3.5}$$

which clearly does work. Infact this is doing the same thing as described in Fig 16. That is; we are stretching the phase space in one direction and squeezing it by an *equal* amount in the other direction and hence conserving the area and this obeys Liouville's theorem.

Another slightly more complex example of transformations is:

$$\begin{aligned} P &= p\cos\theta + q\sin\theta \\ Q &= -p\sin\theta + q\cos\theta \end{aligned} \tag{5.3.6}$$

This is a form of rotation in phase space. Computing the Poisson bracket:

$$\{Q, P\} = \{q\cos\theta - p\sin\theta, p\cos\theta + q\sin\theta\} = \cos^2\theta + \sin^2\theta = 1 \tag{5.3.7}$$

So this transformation also conserves the Poisson bracket structure. In general, the transformations that conserve the Poisson bracket structure are called *canonical* transformations.

If we can build up all the transformations from infinitesimal transformations, where infinitesimal transformation means:

$$Q_i = q_i + \delta q_i(q_i, p_i) \tag{5.3.8}$$

(the point of adding an infinitesimal term is that we can drop higher order terms), then we can define canonical transformations as:

$$\{Q, P\} = \{q, p\} + \{\delta q, p\} + \{q, \delta p\} \tag{5.3.9}$$

Note that I have left out a term, $\{\delta q, \delta p\}$ as that is very small and therefore neglected. We want the Poisson bracket structure to be conservd, i.e:

$$\{Q, P\} = \{q, p\} \tag{5.3.10}$$

This means that we require:

$$-\{\delta q, \delta p\} = \{q, \delta p\} \tag{5.3.11}$$

And this is what defines a canonical transformation. Suppose:

$$\delta q = \epsilon\{q, G(p, q)\} \tag{5.3.12}$$

where $G(p, q)$ is called the generator of the canonical transformation and similar expression for p:

$$\delta p = \epsilon\{p, G(p, q)\} \tag{5.3.13}$$

We claim that if δp and δq are obtained using the generators, then the Poisson bracket structure is conserved. To prove this lets compute:

$$\{q, G(p, q)\} = \frac{\partial G(p, q)}{\partial p} \tag{5.3.14}$$

Inserting this into Eq 5.3.12:

$$\delta q = \epsilon\frac{\partial G(p, q)}{\partial p} \tag{5.3.15}$$

Similar for δp:

$$\delta p = -\epsilon \frac{\partial G(p,q)}{\partial q} \tag{5.3.16}$$

Substitute this into one the definitions:

$$\{p, \delta q\} = -\{\delta p, q\} \tag{5.3.17}$$

$$\begin{aligned}
\epsilon\{\frac{\partial G(p,q)}{\partial p}, p\} &= \epsilon\{q, \frac{\partial G(p,q)}{\partial q}\} \\
\epsilon \frac{\partial G(p,q)}{\partial p \partial q} &= \epsilon \frac{\partial G(p,q)}{\partial q \partial p}
\end{aligned} \tag{5.3.18}$$

This completes the proof that δp's and δq's that come from generators maintain the Poisson bracket structure. Infact, we have found that flows created by Poisson bracketting with respect to a generator. always defines a canonical transformation.

A special case of this is the Hamiltonian flow. All canonical transformations can be created by choosing an appropriate generator (it is as if in our mind, we could image a Hamiltonian that cause a flow that took points in phase space from one point to another). Infact G could just be the Hamiltonian for another system!.
If we do a canonical transformation on a system, what is the subclass of transformations that are called symmetries?

The answer is; symmetries are the transformations that do not change the Hamiltonian(energy). The best way to make sense of this is by looking at the phase space:

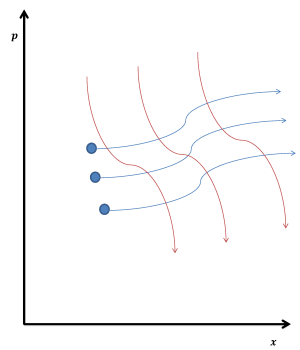


Figure 19: Flows in phase space

Here the red lines represent the flow coming from the generator and the blue lines represent the flow coming from the Hamiltonian. In general, the flow coming from the generator will affect the flow coming from the Hamiltonian, which will change the energy. The only time the energy will not change is if the generator flow will move the system along lines of constant energy in phase space and this is what is called a *symmetry*.

Lets take a general function of q and p and see how it changes along the flow due to the generator:

$$\delta A = \frac{\partial A}{\partial p}\delta p + \frac{\partial A}{\partial q}\delta q \tag{5.3.19}$$

Substitute for δp and δq using Eq 5.3.15 and Eq 5.3.16:

$$\delta A = -\left(\frac{\partial A}{\partial p}\frac{\partial G}{\partial q} + \frac{\partial A}{\partial q}\frac{\partial G}{\partial p}\right)\epsilon = \epsilon\{A, G\} \tag{5.3.20}$$

This is exactly the same formula that we used for the Hamiltonian, when G was the time derivative. So for the energy to be conserved, we get the condition:

$$\{H, G\} = 0 \tag{5.3.21}$$

As it says that H does not change along the flow G. Infact we can flip the equation:

$$\{G, H\} = 0 \tag{5.3.22}$$

and say that G does not change with time. So G itself is a conserved quantity (remember every symmetry has a conserved quantity associated with it, so this is intuitively pleasing).

As an example, lets consider the Hamiltonian of a free particle in 2D:

$$H = \frac{p^2}{2m} = \frac{(p_x^2 + p_y^2)}{2m} \tag{5.3.23}$$

and the angular momentum is:

$$L_x = xp_y - yp_x \tag{5.3.24}$$

Lets suppose that the angular momentum is the generator in this case, $L_x = G$:

$$\begin{aligned}
\{G, H\} &= \{xp_y - yp_x, \frac{(p_x^2 + p_y^2)}{2m}\} \\
&= \{xp_y, \frac{p_x^2}{2m}\} - \{yp_x, \frac{p_y^2}{2m}\} \\
&= p_y\{x, \frac{p_x^2}{2m}\} - p_x\{y, \frac{p_y^2}{2m}\} \\
&= p_y p_x - p_y p_x \\
&= 0
\end{aligned} \tag{5.3.25}$$

6 Deterministic laws and the need for Quantum mechanics*

To conclude I would like to briefly talk about an idea that has bothered me for a long time, and since I have started studying physics my idea has become even more puzzling.

Until the 19^{th} century it was believed that all the laws of physics had been worked out [9]. Now I shall present an argument that will lead to profound consequences on the assumption of those laws. If nature really obeys Newtonian mechanics and statistical mechanical models of thermodynamics, then consider the "‘beginning"’ of the universe (I shall not discuss the what the beginning was, that is discussion for another time!), whatever/wherever it was. All the particles will have had a certain position in space and would have been subject to certain forces, which were believed to be Newtonian mechanics and thermodynamics.

The most fundamental concept in these laws, is that they are deterministic and this is what I have been stressing all through this course. But if this is really how nature works, then at that very point when the universe "‘began"’ surely everything had to have been decided, in terms of how the universe was going to evolve as $t \to \infty$, which would surely mean that we cannot influence anything in the future, not even *our* future! This would mean that the concept of free-will is an illusion, which itself would lead to extremely serious permutations in all walks of everyday society. For example the law system, a criminal could claim that what he did, was not actually his fault, it was decided at the beginning of the "‘universe"’. I fully understand how speculative this idea is, and that it is probably not an argument to be had as part of a physics course, however I feel that it is very important (although I doubt it will help anyone pass their classical mechanics exams!).

I have to stress that the argument that we can never measure something to infinite precision due to our equipment not being completely predictable, does not affect the point I am trying to make. The point is that it doesnt matter weather *we* can predict the future; nature has itself has already decided what will happen in the future. I suppose this leads to the question about *why* did the universe begin the way it did. This is a question that most physicists never go anywhere near, infact whenever a fundamental problem is questioned by the term *why*, the answer becomes unknown but again this is another statement one could discuss for hours so its not something I could write about here. Returning to the point I am making, it seems as if the deterministic laws even though seem obvious to us in our everyday life, if thought about carefully lead to remarkable problems.

[9] as quoted by Lord Kelvin

Another consequence of these deterministic laws is that they would mean natural processes like Darwinian evolution, which are fundamental dependent on randomness in nature, would no longer be considered random. It almost as if they would have a purpose and I think this is an argument that would certainly light up people with a creationist view as if there is a "'creator"' who created the universe than indeed he would be the one responsible for everything that happened in the universe in accordance with these deterministic laws.

In my mind this leads to a need for LOM that are not completely deterministic. Indeed these were provided in the form of Quantum mechanics (QM) in the 20^{th} century with the work of Einstein, Schrodinger, Heisenberg etc. The laws of QM are *probabilistic* (of course non deterministic laws do not have to follow laws of probability, they could be completely random, but probabilistic laws are certainly better than deterministic laws if one wants to overcome the argument that I presented) as supposed to deterministic as nature has an inherent uncertainty hidden in it that is described by the famous Heisenberg uncertainty principle. It is important to emphasise that this uncertainty does not originate from imprecision in measuring equipment or anything, it is always there in nature.

As an example, suppose there is a particle which only has a property of spin. Suppose it is prepared with a particular spin, and try to measure it using an apparatus (what the apparatus is and how a spin is prepared is not important here). Even though a spin is prepared in a particular state, there is no way to say, *a priori*, what the outcome of the measurement would be. Quantum systems exist in states of superposition and "'collapse"' into a particular state when a measurement is made and there is no way of determining what the outcome will be, one can simply make probabilistic statements. In this example of the spin, all we can say is that the spin can either be "'up"' or "'down"' with a 50/50 chance (assuming there are no external affects like magnetic fields). These probabilistic LOM would provide a way out of the argument that everything was decided in the beginning as we have no way of determining what an experiment would observe. But this does not completely solve the problem, I think, because we cannot tell what will happen in an experiment with deterministic law as-well when system are chaotic, such as weather systems, so what exactly is the big improvement? A coin that is flipped has a 50/50 chance of being heads or tails and it does simply works under the LOM of classical mechanics (inherently everything will of course be QM), but I believe the improvement from in QM is that the laws are *intrinsically* random as supposed being random due to the complexity of a system.

All of this is just my opinion and anyone who would think otherwise will also be equally valid, however I feel that the fact the deterministic laws lead to such problems means that nature would inherently be non-deterministic and therefore Newton's laws and classical mechanics could not have possibly been the final answer and should have been a signal for physicists in during this time to keep looking for how nature really worked [10] and hence the need for Quantum mechanics.

[10] I don't know weather any physicists did infact think of this and realise that the laws of nature could be deterministic and kept searching for what we now know as Quantum mechanics